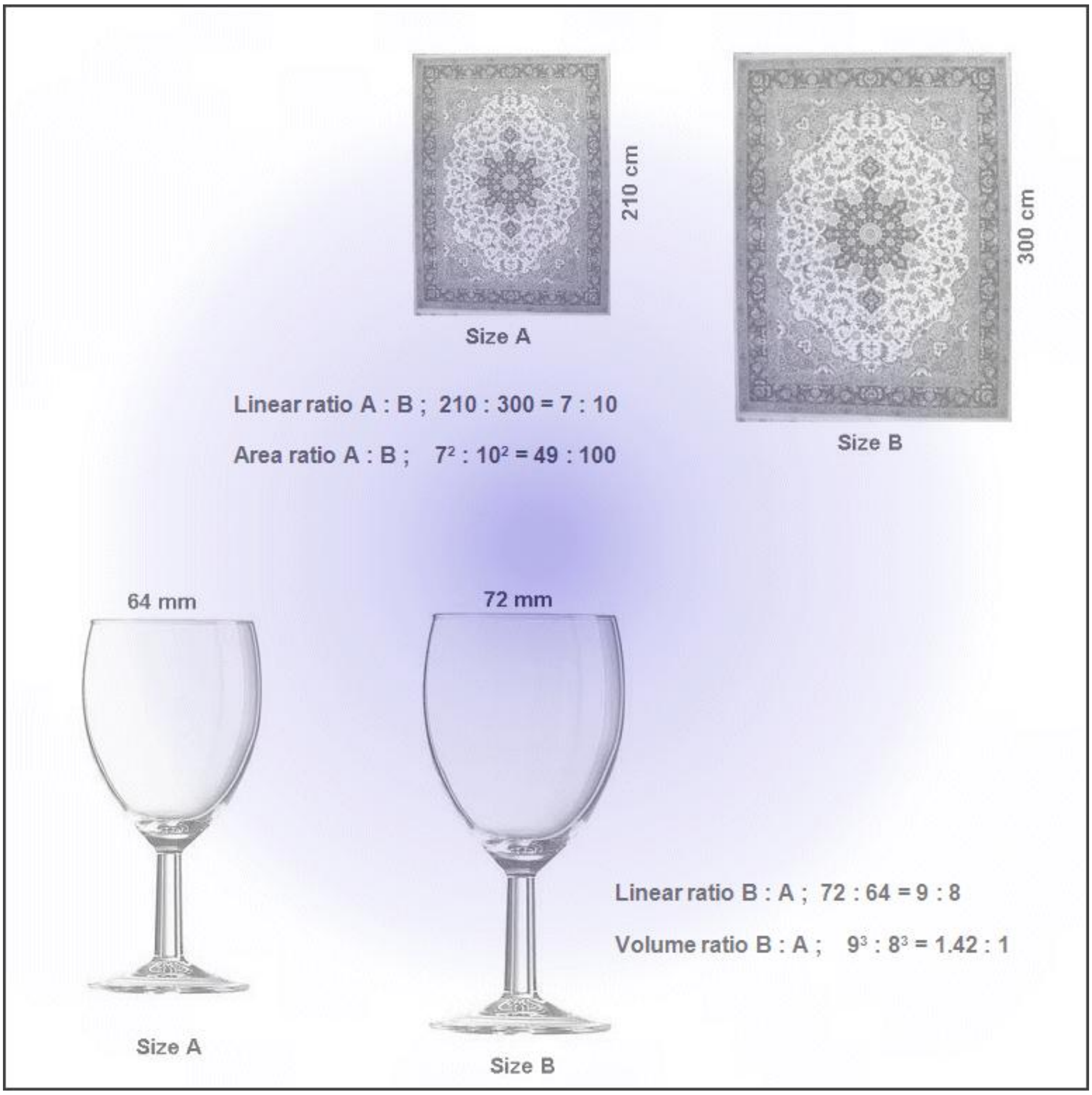


M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

AREAS, VOLUMES AND SIMILARITY



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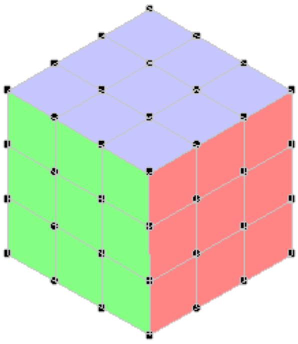
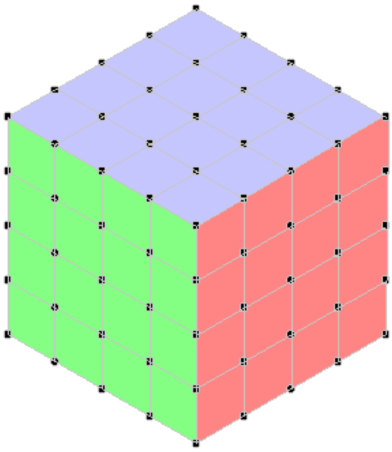
AREAS, VOLUMES AND SIMILARITY

We begin this session with an investigation.

The Rubik Cube and its successor, Rubik's Revenge were popular toys (and forms of mental torture !) to many people a generation ago. The original cube had three small cubes sharing each edge, whereas Rubik's Revenge had four. Both puzzles employed individual small cubes of the same size.
How, therefore, were the lengths, areas and volumes of Rubik's Cube and Rubik's Revenge related ?



Rubik's Cube, 1981



Rubik's Revenge, 1982

We can see at once that the lengths of the edges of the two large cubes are related in the ratio 3:4.

To find the relationship between the surface areas of the two large cubes, we take a look at a single square face of each. A face of the original cube is made up of 3×3 or 9 small squares, but one face of Rubik's Revenge is made up of 4×4 or 16 small squares.

$\therefore$ The ratio of the areas of the two large cubes is 9:16.

Finally, we look at the ratio of the volumes: the original cube is made up of $3 \times 3 \times 3$ or 27 small cubes, but Rubik's Revenge is made up of $4 \times 4 \times 4$ or 64 of them.

$\therefore$ The ratio of the volumes of the two large cubes is 27:64.

The general rules follow on from this specific example.

If the linear dimensions of two similar figures are in the ratio a : b, then

- **their areas will be in the ratio $a^2 : b^2$**
- **their volumes will be in the ratio $a^3 : b^3$**

Example (1):
A home-furnishing firm imports rugs for domestic floors.

One rug in the range is available in two sizes.

Given that the rugs illustrated are similar, show that the smaller rug has an area just less than half that of the larger one.

The long sides of the two rugs correspond, so the *linear* dimensions of rug A are $\frac{210}{300}$ or $\frac{7}{10}$ those of rug B.

The *area* of rug A is therefore $\left(\frac{7}{10}\right)^2$ or $\frac{49}{100}$, i.e. 49% of

the area of rug B, i.e. just under half the area of rug B.

Example (2): A glassmaking firm makes two sizes of similar wine glass.

The smaller glass has a capacity of 125ml.

Show that the larger glass has a capacity of about 175ml.

The diameters of the rims of the glasses are in the ratio 72:64.

Capacity is the same as volume, so the ratio of the volumes of the glasses is therefore $72^3:64^3$ or approximately 1.42:1.

The capacity of the larger glass is therefore (125×1.42) or 178ml, which is close to 175ml as requested.

Example (3): Ceramic tiles are sold in packs of 80 to cover a given area. Each tile is square of side 150mm. The manufacturer also sells larger square tiles in packs of 20 to cover the identical area. What is the side length of the larger square tile ?

Because 20 is one quarter of 80, and the two packs cover the same area, it follows that 80 small tiles have the same area as 20 large ones.

The **areas** of the tiles are therefore in the ratio 1: 4, so we have to take square roots to find the corresponding ratios of the lengths, namely 1:2.

The side length of the larger tile is double that of the smaller one, namely 300mm.

Check (millimetres have been converted to metres)

$80 \times 0.15 \times 0.15 = 1.8$ $\therefore$ 80 smaller tiles cover 1.8 square metres.
$20 \times 0.3 \times 0.3 = 1.8$ $\therefore$ 20 larger tiles cover 1.8 square metres.

Example (4): The interior sets for Bilbo Baggins' hobbit-hole at Bag End are built to two different scales for the *Lord of the Rings* and *Hobbit* films; full size and two-thirds linear full size.

Aside from that, all the fixtures and fittings in the two sets are mathematically similar.

i) The diameter of the door on the reduced-size set is 1.12 m. What is the corresponding diameter of the door in the full-size set ?

ii) Painting the interior walls and ceiling of the full-size set required 45 litres of paint. How much paint was used for the corresponding surfaces of the reduced-size set ?

iii) The capacity of the kettle in the reduced set is 400 ml. Calculate the capacity of the kettle in the full-size set.

i) The ratio between the *linear* dimensions of the two hobbit-holes is 2 : 3, and therefore the diameter of the door in the full-size set is $\frac{3}{2} \times 1.12$ m , or **1.68m**.

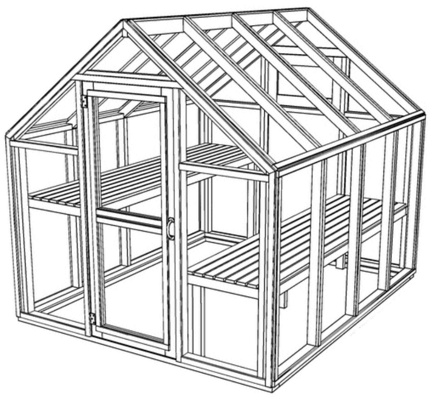
ii) When painting, we are dealing with *area*, and so the ratio between the painted areas of the two hobbit-holes is $2^2 : 3^2$, i.e. 4 : 9.

Hence the amount of paint used for the reduced-size hobbit-hole was $\frac{4}{9} \times 45$ litres, or **20 litres**.

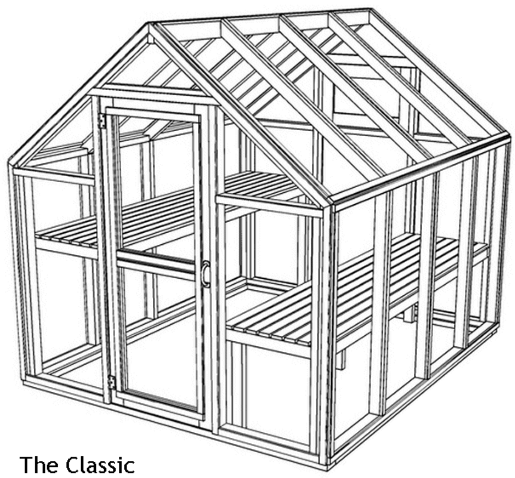
iii) Capacity is the same as *volume*, and so the ratio between the volumes of the two kettles is $2^3 : 3^3$, or 8 : 27.

The capacity of the kettle in the full-sized set is thus $\frac{27}{8} \times 400$ ml or **1350 ml**, i.e. **1.35 litres.**

Example (5): The following two greenhouses are mathematically similar.



The Compact
Maximum Height 2.0 m



The Classic
Maximum Height 2.4 m

i) The volume of the Compact greenhouse is 8.75 m^3. Calculate the volume of the Classic greenhouse.

ii) The total surface area of the glass in the Classic greenhouse is 28.8 m^2. Calculate the total glass surface area of the Compact greenhouse.

We have a ratio of 2.0 : 2.4 between the maximum heights of the two greenhouses, which can be simplified to 5 : 6.

i) The heights are related in the ratio 5 : 6 , therefore the corresponding ratio for the volumes is $5^3 : 6^3$, or 125 : 216.

$\therefore$ The larger Classic greenhouse has a volume of $\frac{216}{125} \times 8.75 \, \text{m}^3$, or **15.12 m^3**.

ii) The areas of the two greenhouses are related in the ratio $5^2 : 6^2$, or 25 : 36.

$\therefore$ The smaller Compact greenhouse has a total glass surface area of $\frac{25}{36} \times 28.8 \, \text{m}^2$, or **20 m^2**.

Example (6): (Percentage revision !)

A confectionery company offers a seasonal chocolate box in the shape of an octagonal prism.

This particular box contained 1.2 kg of chocolates in 2011, but the company had decided to reduce all linear dimensions annually by 3% over the following four years, whilst keeping the boxes mathematically similar.

What was the weight of the chocolates in the box in 2015, assuming that the packing and weight / volume ratios remained unchanged ?

Give your answer to the nearest 10 grams.

Firstly, we must realise that a reduction of 3% in *linear* dimensions corresponds to a multiplier of 0.97, which needs to be cubed to convert it into a *volume* multiplier of 0.97^3, or 0.913 to 3 decimal places.

Secondly, we have to apply this multiplier of 0.97^3 over a time span of four years, and so the final volume multiplier is $(0.97^3)^4$, or 0.97^{12} = 0.694 to 3 decimal places, using the laws of indices.

We are assuming that the weight / volume ratios had remained unchanged, so the final weight of the chocolates in the 2015 box is (1200×0.694) g, or **830 g** to the nearest 10 grams.