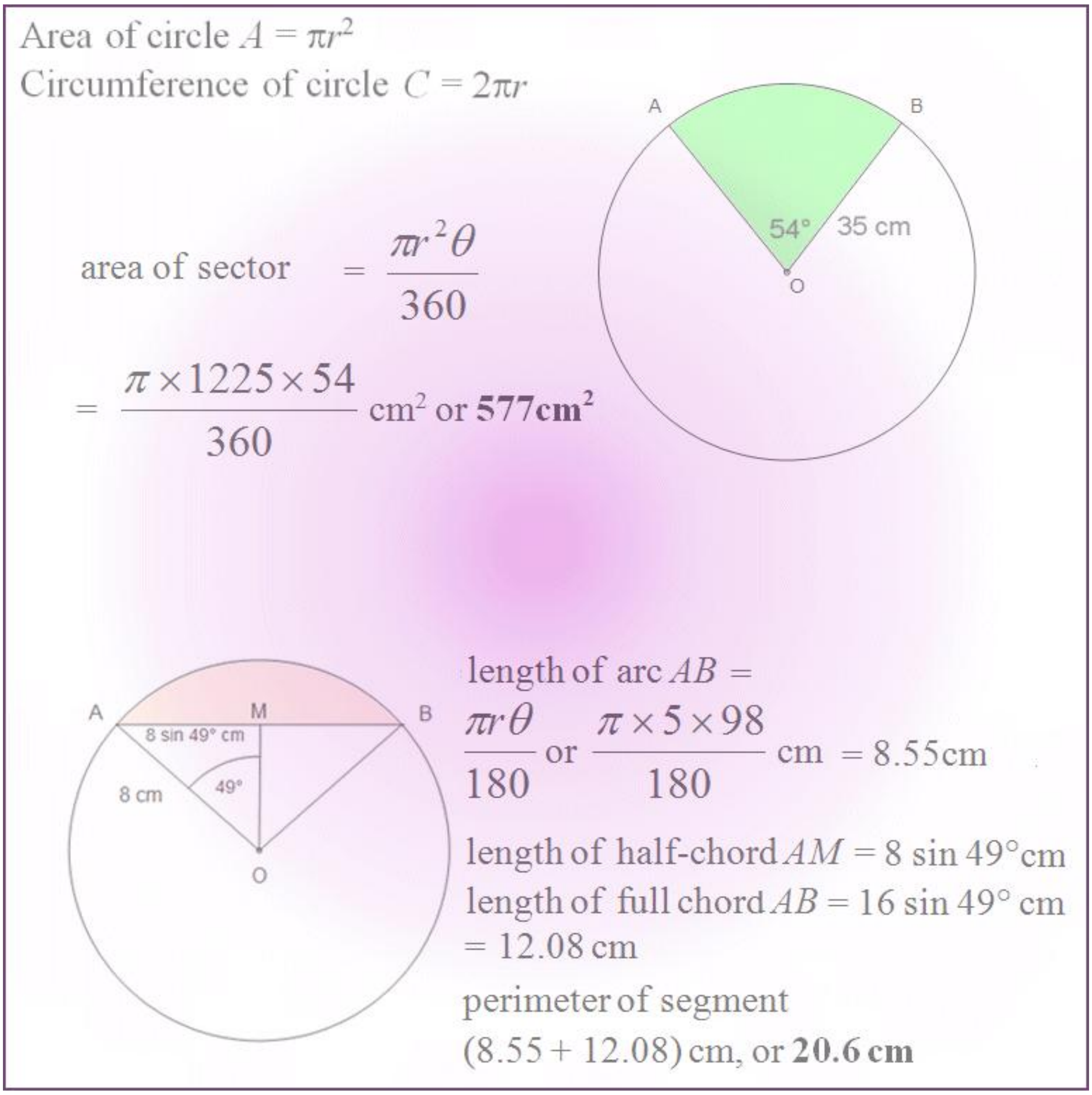
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

CIRCULAR MEASURE - SECTORS AND SEGMENTS



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CIRCULAR MEASURE

Recall the following formulae:

- Area of a circle is $A = \pi r^2$ where r is the radius.
- Circumference of a circle is $C = 2\pi r$ where r is the radius (or πd where d is the diameter).

For revision of simpler examples, check out "Circles" on the Foundation Tier study notes.

Example (1): A bicycle has a rev counter on its front wheel counting the number of revolutions made from a starting point.

A cyclist resets the rev counter to zero before a trial race of 4km. The rev counter reads 1852 at the end of the trial. What is the diameter of the front cycle wheel ? (Use the π key on your calculator).

Here, 1852 revolutions = 4000m, so one revolution = 1 circumference = $\frac{4000}{1852}$ m or 2.160m

$\therefore$ Diameter of wheel = $\frac{2.160}{\pi}$ m = 0.687m = 69cm to nearest cm.

Example (2): A circular ornamental lake in a park has a diameter of 38m. Work out its area, using the π key on your calculator.

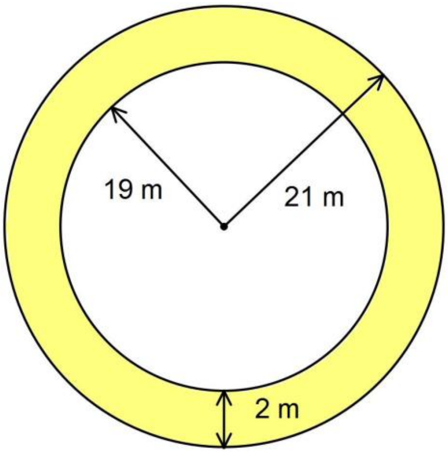
We are given the diameter here, so we need to halve it first to obtain the radius, namely 19m.
The area is πr^2 , or $\pi \times 19^2$, or 361π m^2 or 1130 m^2 to 3 s.f.

Example (3): The park authorities in Example (2) have decided to lay a gravelled path 2 metres wide all around the circumference of the lake.

i) Show that the total area of the gravelled path is 80π square metres.

ii) Given that the depth of gravel in the path is 10 cm, and that the material and labour costs are £179 per cubic metre, what is the cost of laying the path ?

(Use exact amounts, without rounding to the nearest cubic metre above. Give the answer to the nearest £.)



i) The area of the lake is 361π m^2 from the previous question.
Adding the width of the path to the lake's radius, we have a larger circle with a radius of 21 m.

The area of this larger circle is $\pi \times 21^2$, or 441π m^2, so, by subtraction, the area of the path is $(441 - 361)\pi$ m^2, or 80π m^2.

Hint: Whenever a circle of radius r is entirely inside a larger circle of radius R, the area of the ring so formed is $\pi(R^2 - r^2)$.

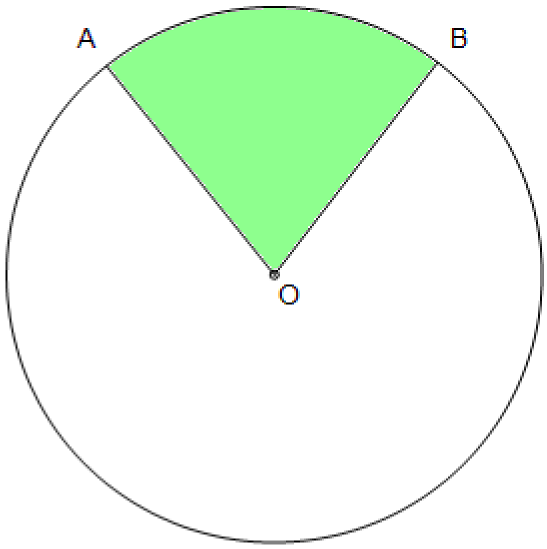
ii) If the depth of gravel in the path is 10 cm, or 0.1 m , then the volume of gravel used is $80\pi \times 0.1$ m^3 or 8π m^3.

$\therefore$ The cost of laying the gravel path is £ $(179 \times 8\pi)$ **= £4499.**

Areas and arc lengths of sectors and segments.

Recall the definitions of a sector and a segment.

The sector.



A sector is a part of a circle bounded by two radii and an arc. In the diagram, *OA* and *OB* are radii and *AB* is the corresponding arc.

The shaded region is the sector *AOB.*
In addition, the region is a minor sector because its area is less than half that of the circle, and the clockwise arc *AB* is a minor arc.

The unshaded region is a major sector since its area is more than half that of the circle, and the anticlockwise arc *AB* is a major arc.

A whole circle can be said to be a sector with an angle of 360°, and its area is
$A = \pi r^2$.

It can be seen that if a sector has an angle $\theta°$, this angle is $\frac{\theta}{360}$ of a circle.

We can therefore find the area of the sector by multiplying the area of the circle by $\frac{\theta}{360}$.

The area of a sector of angle $\theta°$ and radius r is therefore $\frac{\pi r^2 \theta}{360}$.

Similarly, the arc length of a sector
can be obtained from the formula for the circumference and multiplying it by $\frac{\theta}{360}$.

$\therefore$ arc length of a sector of angle $\theta°$ and radius r is given by $\frac{(2\pi r)\theta}{360} = \frac{\pi r \theta}{180}$.

Example (4): Find the area of a 54° sector of a circle of radius 35 cm.

We substitute $\theta = 54°$ and $r = 35$ cm into the formula $\frac{\pi r^2 \theta}{360}$ to obtain an area of the sector of

$\frac{\pi \times 1225 \times 54}{360}$ cm^2 or **577cm^2.**

Example (5): A semicircular table top has a straight edge of 1.26 m. Calculate its area and perimeter, giving answers to 3 significant figures.

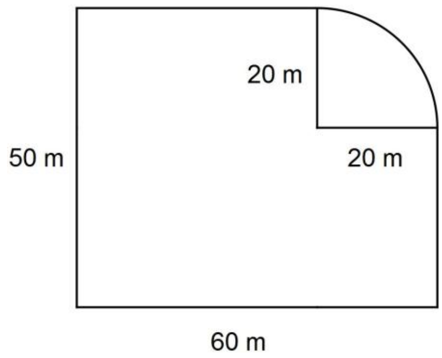
A semicircle can be said to be a sector whose angle is 180°, or half a circle, so the formula for the area simplifies to $\frac{\pi r^2}{2}$ and, for the arc length, πr.

The straight edge of the table top is its diameter, so the radius is 0.63 m.

The area is $\frac{\pi r^2}{2} = \frac{\pi(0.63^2)}{2}$ = 0.623 m^2.

The perimeter is $\pi r + 2r = (0.63\pi + 1.26)$ m = 3.24 m .

Example (6): A school's playing field has the following plan, where the section in the top right-hand corner is a quarter-circle. Find its area to the nearest square metre and perimeter to the nearest metre.



The playing field is basically a rectangle measuring 60 m by 50 m, but with a square of side 20 m removed and replaced with a quarter-circle of radius 20 m.

A quarter-circle is a sector whose angle is 90°, so its simplified area formula is $\frac{\pi r^2}{4}$ and, for the arc length, $\frac{\pi r}{2}$.

Therefore, area of original rectangle = 60 m × 50 m = 3000 m^2.
Area of square (to be subtracted) = 20^2 m^2 = 400 m^2.

Area of quarter-circle (to be added) = $\frac{\pi(20)^2}{4}$ = 100π m^2 = 314 m^2.

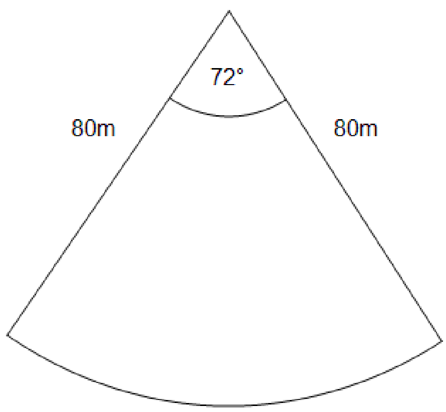
Hence the area of the playing field = (3000 – 400 + 314) m^2 = **2914 m^2.**

The straight line sections of the playing field total 60 + 50 + (60 – 20) + (50 – 20) m, or 180 m.

The quarter-circular arc is $\frac{\pi r}{2}$ = 10π m, or 31 m to the nearest metre.

The perimeter of the playing field is 180 + 31 = **211 m** .

Example (7): An enclosure is in the shape of a 72° sector of a circle of radius 80m.
Find the cost of fencing the enclosure, where fencing costs £16 per metre.



We need to find the perimeter here; it is twice the radius plus the arc length.

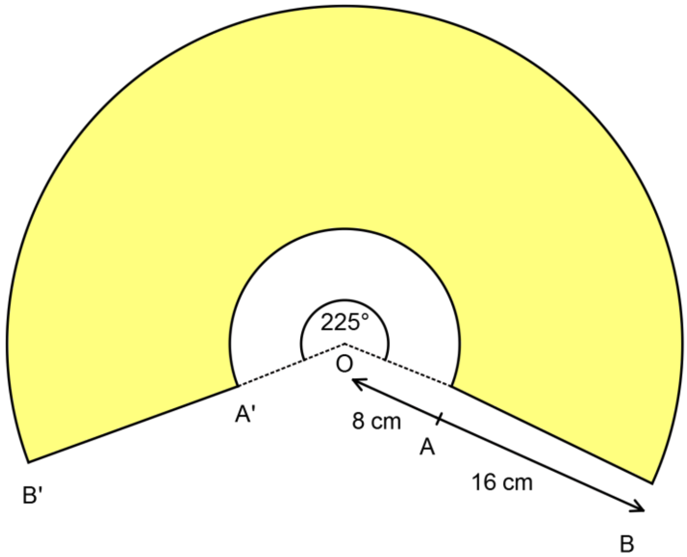
The arc length is $\frac{\pi \times 80 \times 72}{180}$ m or 100.5m.

Twice the radius is 160m, so the perimeter is 261m to the next metre above.
The cost of fencing the enclosure is £16 × 261, or **£4176**.

Example (8): The piece of plastic shown on the right is shaped to form a lampshade by connecting point *A* to *A'* and *B* to *B'*. It consists of a 225° sector of a circle with a smaller sector removed .

The distance from the centre of the sector *O* to the 'inside' of the plastic piece *A* is 8 cm; also, the distance *AB* from the 'inside' to the 'outside of the piece is 16 cm.

Calculate, to the nearest square centimetre, the area of the piece of plastic needed to make the lampshade.



The area of a sector is given by $\frac{\pi r^2 \theta}{360}$ where *r* is the radius and θ is the angle.

The area of the plastic is that of the sector of radius *OA* subtracted from that of the sector of radius *OB.*

Let *R* be the radius *OB* of the larger sector : it is (8 + 16) cm, or 24 cm.
Let *r* be the radius *OA* of the smaller sector (to be subtracted) : it is 8 cm.

The angle θ is 225°, so $\frac{\theta}{360} = \frac{225}{360} = \frac{5}{8}$, i.e. the sector is equivalent to $\frac{5}{8}$ of a circle.

The area of the difference, i.e. that of the plastic piece, is $\frac{\pi R^2 \theta}{360}$ - $\frac{\pi r^2 \theta}{360}$ or, more concisely, $\frac{\pi \theta}{360}\left(R^2 - r^2\right)$. (See Example (3) for a similar case.)

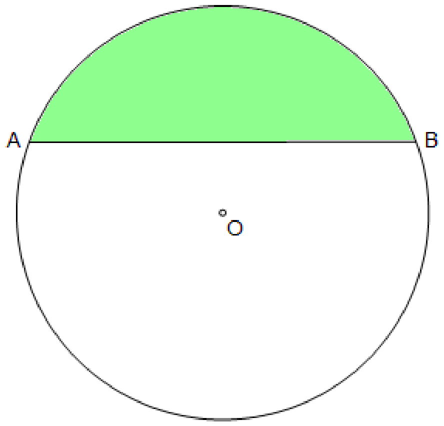
Substituting for *R*, *r* and θ , the area of the plastic is

$$\frac{5\pi}{8}\left(24^2 - 8^2\right) = \frac{5\pi}{8}\left(576 - 64\right) = \frac{5\pi \times 512}{8} = 320\pi \text{ cm}^2,$$ or **1005 cm²**.

The segment.

A segment is a part of a circle bounded by a chord and an arc. In the diagram, the line *AB* is the chord, and the curve *AB* is the corresponding arc. (The chord is not a diameter, since it does not pass through the centre *O*.)

The shaded region is the segment *A*, and it is a minor segment because its area is less than half that of the circle. Conversely, the unshaded region is a major segment since its area is more than half that of the circle.

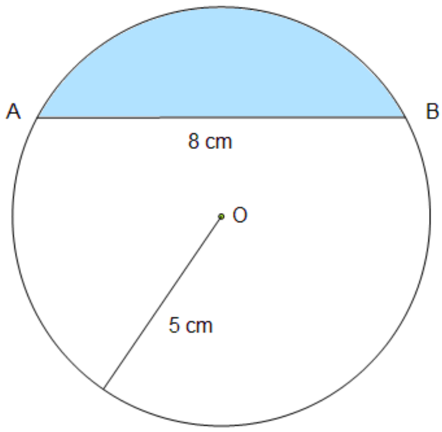


Finding the area and arc length of a segment.

Segments are a little more awkward to work with than sectors, usually requiring the use of trig.

Example (9): Find the perimeter and area of the segment shown.

We are given the length of the chord *AB* and the radius, but not the angle *AOB*.
To find this angle, we need to construct the isosceles triangle *AOB* and use trig methods.

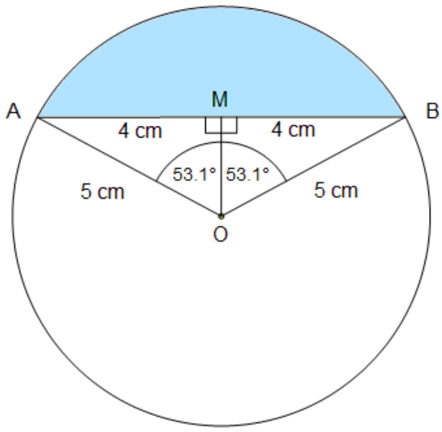


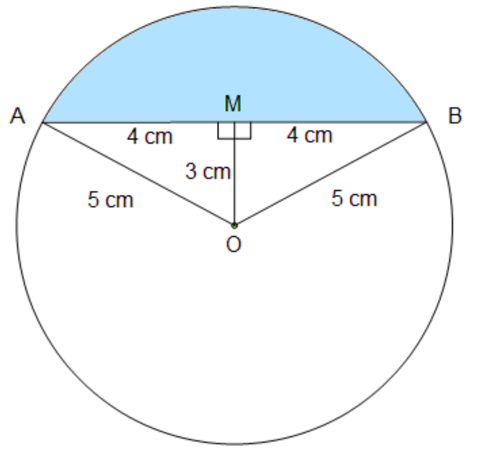
The midpoint *M* of the chord *AB* divides the isosceles triangle *AOB* into two congruent right-angled triangles *AMO* and *BMO.*

Using trig, $\sin \angle AOM = \frac{AM}{OA} = \frac{4}{5}$, giving $\angle AOM$ = 53.1°.

Since $\angle AOM = \angle BOM$, $\angle AOB = 2(\angle AOM)$ = 106.3°.

The length of arc *AB* is therefore $\frac{\pi \times 5 \times 106.3}{180}$ cm or 9.3 cm,
hence the perimeter of the segment is 8 + 9.3 cm, or **17.3 cm.**





The best way of working out the area of a segment is to treat it as a sector with a triangle removed.

To find the area of the triangle *AOB*, we know that the base *AB* = 8 cm, but we need to find the height *OM*.

For that, we use Pythagoras on the triangle *AMO*, whereby $(OM)^2 = (AO)^2 - (AM)^2$, giving *OM* = 3 cm.

The area of triangle *AMO* is ½ × 8 × 3, or 12 cm^2.

From the earlier result, the area of the sector *AOB* is

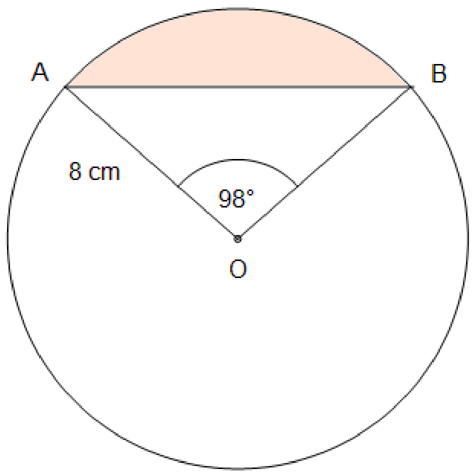
$$\frac{\pi r^2\theta}{360} \text{ or } \frac{\pi\times 25\times 106.26}{360} \text{ cm}^2 \text{ or } 23.2 \text{ cm}^2.$$

Therefore:

Area of sector = 23.2 cm^2
Area of isosceles triangle = 12.0cm^2

Area of segment by subtraction = 11.2 cm^2.

Example (10): Find the area and perimeter of the segment shown on the right.



The example is considerably simpler than the previous one as we are already given the angle subtended by the segment.

The area of the sector can be worked out at once as

$$\frac{\pi r^2\theta}{360} = \frac{\pi \times 64 \times 98}{360} \text{ cm}^2 \text{ or } 54.7 \text{ cm}^2.$$

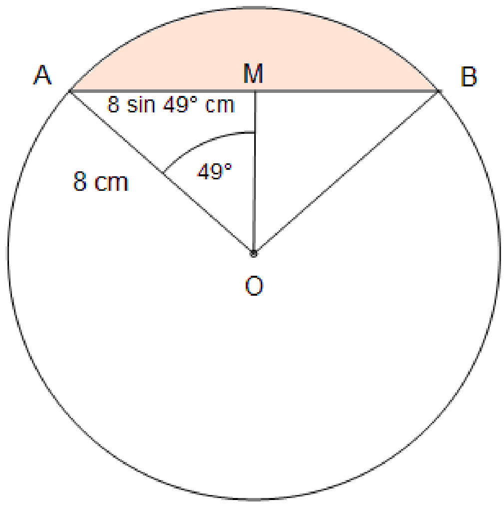
To find the area of the isosceles triangle, we recall the formula $A = ½ab \sin C$, where a and b are two sides and C the included angle.

The two sides are both radii at 8 cm and the included angle is 98°, so the area of the triangle is $½ \times 8^2 \times \sin 98°$ cm^2, or 31.7 cm^2.

∴ The area of the segment (by subtraction) = (54.7 – 31.7) cm^2 = 23.0 cm^2.

Finding the perimeter is a little more tricky.

The length of the arc is easy enough to find, it is $\frac{\pi r\theta}{180}$ or $\frac{\pi \times 5 \times 98}{180}$ cm, or 8.55cm.



We need to use trig to find the length of the chord *AB*.

The isosceles triangle *AOB* can be split into two right-angled ones, where *M* is the midpoint of the chord *AB.*

The length of the half-chord *AM* = 8 sin 49° by trig, The full chord *AB* is therefore 16 sin 49° cm or 12.08 cm.

If a chord subtends an angle θ at the centre, its length is given by 2*r* (sin ½θ) where *r* is the radius of the circle.

(Or, to find the length of a chord, find the sine of half the subtended angle, and multiply it by twice the radius.)

The perimeter of the segment is therefore (8.55 + 12.08) cm, or **20.6 cm**.

(We could also have found the length of the chord c by applying a modified form of the cosine formula to the isosceles triangle, where $r = 8$ and $C = 98°$.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Because a and b are both radii, we can simplify the formula to $c^2 = 2r^2 - 2r^2\cos C$

$\rightarrow c^2 = 2r^2(1-\cos C)$.

$\therefore\ c^2 = 128(1-\cos 98°)$ = 145.8 → c = 12.08 cm.)