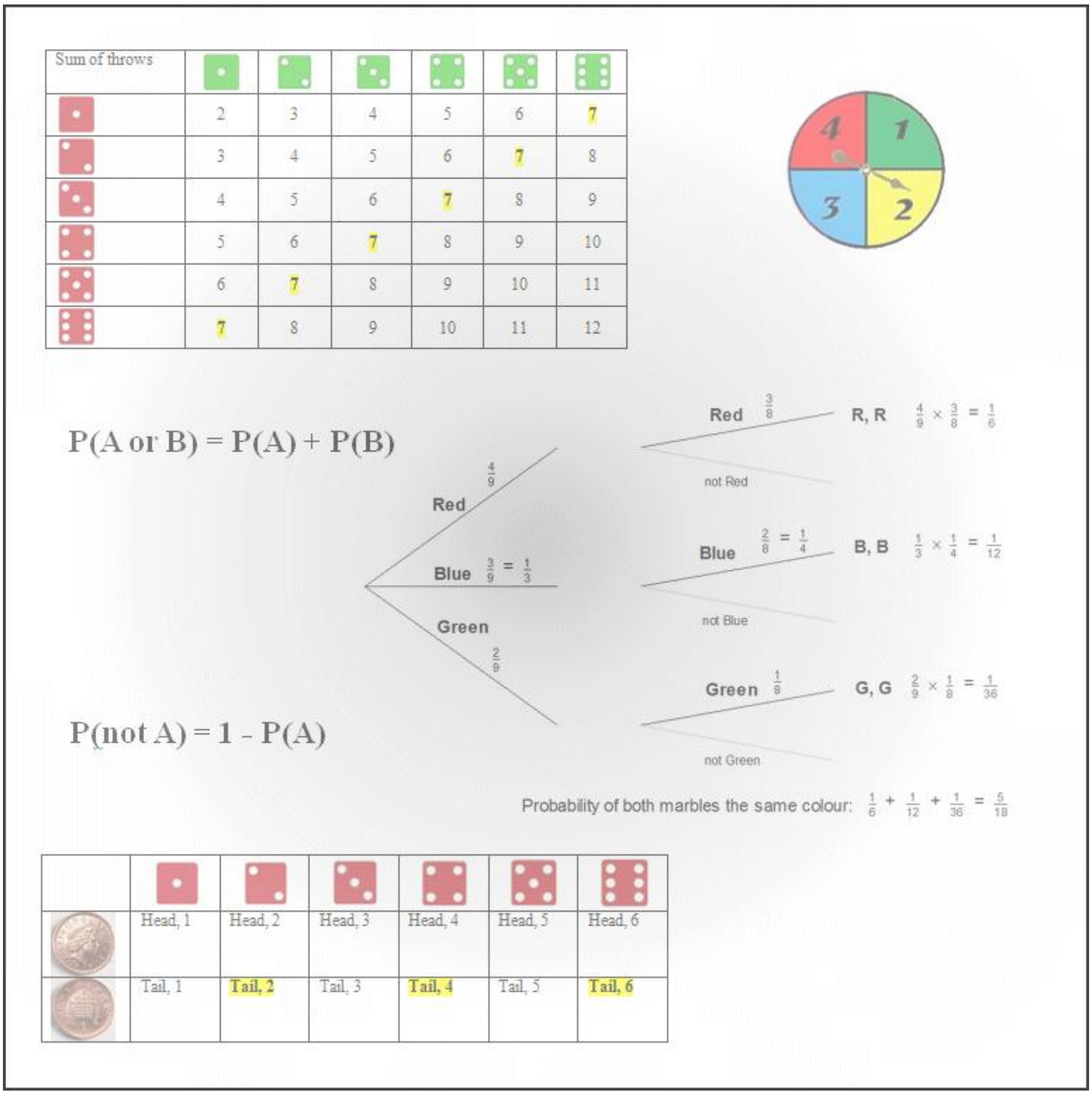
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

PROBABILITY



Version: 3.1 Date: 07-10-2015

PROBABILITY

The probability of the outcome of an event is a fraction between 0 and 1.

A probability of zero means the event cannot occur at all.
A probability of 1 means the event is 100% certain to happen.

Examples of events are tossing a coin, throwing dice, drawing a card from a pack.

The toss of a fair (unbiased) coin has two possible outcomes; 'heads' or 'tails'.
The probability of 'heads' turning up is $\frac{1}{2}$, as is that of 'tails'.
The sum of the two outcomes is 1, and this would also hold good for a biased coin.
If a coin was biased so that 'heads' had a $\frac{3}{5}$ probability of turning up, then 'tails' would have a probability of $\frac{2}{5}$.

Example (1): A fair dice has 6 sides. What are the probabilities of throwing i) a 6, ii) an odd number under 5, and iii) any number except a 3 ?

A dice throw has six possible outcomes, namely 1 to 6, and all are equally likely with a probability of $\frac{1}{6}$. Those outcomes are **mutually exclusive** since it is impossible to throw more than one number at one time.

The probability of throwing a 6, which can be shortened to P(6) is therefore $\frac{1}{6}$.

There are two odd numbers less than 5 on a dice, 1 and 3. Each has a probability of $\frac{1}{6}$ occurring, and so the probability of either occurring is equal to the sum of each, i.e. $\frac{1}{6}+\frac{1}{6}$ or $\frac{1}{3}$.

P(1 or 3) = P(1) + P(3) = $\frac{1}{6}+\frac{1}{6}$ or $\frac{1}{3}$.

In general, if A and B are mutually exclusive events, then the probability of either event occurring can be worked out by adding their individual probabilities together.

P(A or B) = P(A) + P(B).

The probability of throwing a 3, which can be shortened to P(3) is $\frac{1}{6}$.
The result of the throw will either be 3 or 'not 3', and the two probabilities must combine to give 1.
The probability of not throwing a 3 is therefore 1 - $\frac{1}{6}$, or $\frac{5}{6}$.

In general, if A is an event , then the probability of A *not* occurring is the probability of A occurring subtracted from 1.

P(not A) = 1 - P(A).

Biases in probability.

In most of the examples in this section, we will assume objects like coins, spinners or dice to be ‘fair’. By this we mean that the actual probabilities come close to the theoretical ones.

Testing for bias is a separate topic, but usually such tests involve many trials. The greater the number of trials, the more accurate the test. See examples 4a – 4b.

Example (2): Three dice were thrown 600 times each and the results recorded.

	⚀	⚁	⚂	⚃	⚄	⚅
Dice 1	104	98	101	105	95	97
Dice 2	94	104	0	205	101	96
Dice 3	71	95	102	97	103	132

i) One of the dice is mis-spotted. Which one is it ?
ii) Describe the properties of the other two dice.

Because each of the six numbers is equally likely on a dice throw, its probability is $\frac{1}{6}$ and therefore each number would be expected to turn up $\frac{1}{6}$ of 600 times, or 100 times.

i) Dice 2’s results show normal results for four of the numbers, but a result of 3 does not occur even once, whilst 4 occurs twice as often as it should. The dice is mis-spotted, with the 4 on two faces and the 3 absent.

ii) The results for Dice 1 are close to the theoretical (they only vary by a few each way), and so this dice can be passed as fair.

Dice 3’s results show that 1 occurs less often, and 6 occurs more often, than they should do. A discrepancy of about 5 in 100 is acceptable, but not one of about 30. This dice is loaded so as to favour a score of 6.

Relative Frequency.

The dice example in part (3) brings us to the idea of **relative frequency**. This is used to estimate the long-term probability of an event if the dice, coin or spinner is suspected of bias.

If 'heads' were to come up 65 times out of 100 coin tosses, then the relative frequency of 'heads' would be $\frac{65}{100}$ or 0.65, which is some way above the theoretical outcome of 0.5.

Example (3): A dice suspected of bias was thrown 600 times and the results recorded.
Complete the relative frequency table, and from it estimate the expected number of sixes tossed after 2000 throws.

						
Frequency	66	117	101	97	84	135
Relative frequency	0.11	0.195				

The dice was thrown 600 times, so each relative frequency is the actual frequency divided by 600.

The completed table therefore looks like this:

						
Frequency	66	117	101	97	84	135
Relative frequency	0.11	0.195	0.168	0.162	0.14	0.225

There seems to be a strong bias for throwing a 6 and against throwing a 1, and a slightly milder bias for throwing a 2 as opposed to a 5.

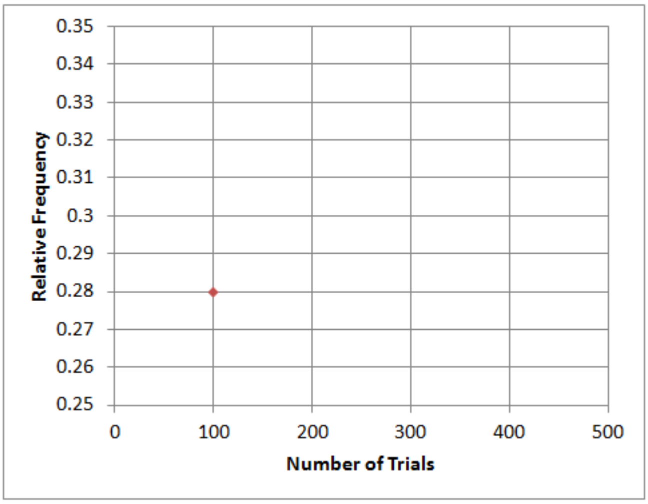
The likely number of sixes to be thrown after 2000 throws is
(relative frequency of a 6) $\times$ (number of throws), here 0.225×2000 or 450.

Example (4a): Jade has been testing a four-sided spinner supplied with a board game. She noticed that a ‘3’ had turned up 28 times after 100 spins, and comes up with this conclusion :

“The spinner is biased, because one quarter of 100 is 25, and a ‘3’ should have only turned up 25 times with a fair spinner.”

Is her reasoning correct here ?

The outcome here is not especially significant, as a relative frequency of $\frac{28}{100}$ or 0.28 is close enough to the ‘fair’ value of one quarter or 0.25 to fall within experimental error. She would need to take more trials to confirm or reject the bias.



Example (4b): Jade continues her experiment counting the number of ‘3’s after various trials. She obtains 57 ‘3’ s after 200 spins, 99 after 300 spins, 128 after 400 spins and finally 162 after 500 spins.

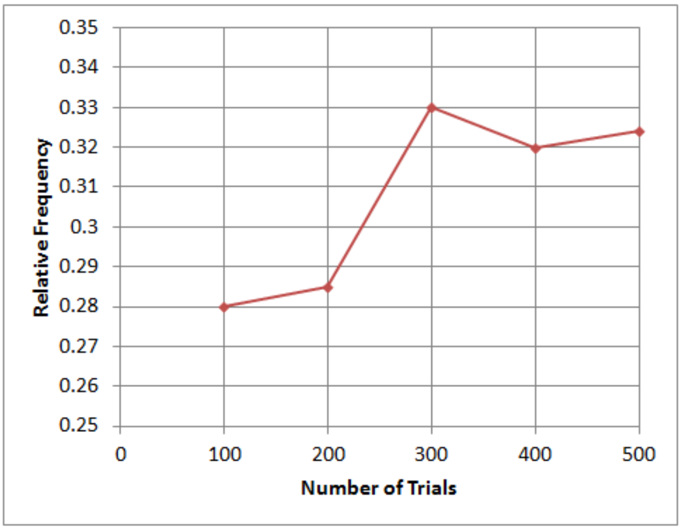
Produce a relative frequency table and use the results to complete the graph on the right.

Is there now a stronger suspicion of the spinner being biased ?

Completing the relative frequency table and graph gives the following:

Number of spins	100	200	300	400	500
Frequency of ‘3’	28	57	99	128	162
Relative freq. of ‘3’	0.28	0.285	0.33	0.32	0.324

The relative frequency of spinning a 3 has not ‘evened’ to a value closer to 0.25, or one quarter, as would be expected of a fair spinner. It appears to have settled to a level of between 0.32 and 0.33, or almost one third, which is a substantial bias.



Back to the sum rule:

If A and B are mutually exclusive events, then the probability of either event occurring can be worked out by adding their individual probabilities together.

P(A or B) = P(A) + P(B).

If A is an event , then the probability of A *not* occurring is the probability of A occurring subtracted from 1.

P(not A) = 1 - P(A).

Example (5): Jack has a 20% probability of a Grade A and a 35% probability of a grade B in his Maths exam. What is the probability of him getting i) either A or B and ii) neither A nor B ?

The two results are mutually exclusive as Jack cannot obtain two different grades in the same exam, and so the sum rule applies.

Calling the probabilities P(A) for a Grade A and P(B) for a Grade B we have

P(A or B) = P(A) + P(B) = 20% + 35% = 55% or 0.55.

The probability of getting neither an A nor a B is therefore 1 – P(A or B) or 0.45.

Independent events.

Two events are said to be independent if the result of one has no bearing on the other.
Thus, if a player was to toss a coin and throw a dice, the result of the coin toss will have no effect on the result of the dice throw.

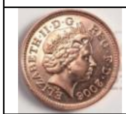
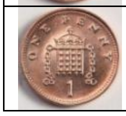
Another example of a series of independent events is a sequence of tosses of an identical fair coin – remember that the coin has no 'memory' of past events.

It is therefore wrong to think on the lines of "We've had tails twenty times, the next toss MUST be a head". The fact that the last twenty tosses resulted in 'tails' makes no difference to the probability a head turning up – it will still be exactly one half.

The possible outcomes can be shown in a **possibility space diagram**.

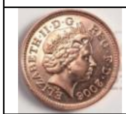
Example (6): A player tosses a fair coin and throws a fair dice.

Draw a possibility space diagram and use it to work out the probabilities of the following events:
i) a head and a number less than 5; ii) a tail and an even number; iii) a head and a 3, or a tail and a 4.

						
	Head, 1	Head, 2	Head, 3	Head, 4	Head, 5	Head, 6
	Tail, 1	Tail, 2	Tail, 3	Tail, 4	Tail, 5	Tail, 6

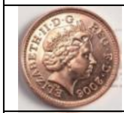
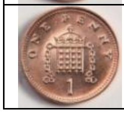
The possibility space diagram shows all the possible combinations of the coin toss and dice throw.
Since there are two possible results of the coin toss and six of the dice throw, there are 2×6 or 12 possible combined results. Also, because both the coin and the dice are fair, each outcome has a probability of $\frac{1}{12}$. (Remember, all probabilities in the space must add to 1).

i) A head and a number less than 5.

						
	**Head, 1**	**Head, 2**	**Head, 3**	**Head, 4**	Head, 5	Head, 6
	Tail, 1	Tail, 2	Tail, 3	Tail, 4	Tail, 5	Tail, 6

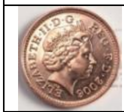
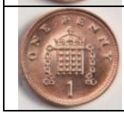
Four of the twelve combinations satisfy the given condition, so its probability is $\frac{4}{12}$ or $\frac{1}{3}$.

ii) A tail and an even number.

						
	Head, 1	Head, 2	Head, 3	Head, 4	Head, 5	Head, 6
	Tail, 1	**Tail, 2**	Tail, 3	**Tail, 4**	Tail, 5	**Tail, 6**

Three combinations satisfy the criteria, so the probability of the event is $\frac{3}{12}$ or $\frac{1}{4}$.

iii) A head and a 3, or a tail and a 4.

						
	Head, 1	Head, 2	**Head, 3**	Head, 4	Head, 5	Head, 6
	Tail, 1	Tail, 2	Tail, 3	**Tail, 4**	Tail, 5	Tail, 6

These two mutually exclusive combinations can have their probabilities added. Each individual event has a probability of $\frac{1}{12}$, so the probability of either event is $\frac{2}{12}$ or $\frac{1}{6}$.

Example (7): Two fair dice are tossed and the sum of their spots recorded.

Draw the possibility space diagram and hence work out the probabilities of the following events:

i) a sum of 7
ii) a 'double' (two equal numbers)
iii) a sum of 6 or 8
iv) a double or a sum of 7
v) a double or a sum of 8

The possibility space diagram looks like this:

Sum of throws						
	**2**	**3**	**4**	**5**	**6**	**7**
	**3**	**4**	**5**	**6**	**7**	**8**
	**4**	**5**	**6**	**7**	**8**	**9**
	**5**	**6**	**7**	**8**	**9**	**10**
	**6**	**7**	**8**	**9**	**10**	**11**
	**7**	**8**	**9**	**10**	**11**	**12**

There are now 6 × 6 or 36 different results of the dice throws, but only 11 possible sums of the spots, namely 2 to 12.

Moreover, not all the sums are equally likely; a sum of 2 can only be obtained in one way (1, then 1) but a sum of 4 can be obtained in three ways :

1 on first dice, 3 on second
2 on first dice, 2 on second
3 on first dice, 1 on second

i) Sum of 7

Sum of throws	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	**7**
⚁	3	4	5	6	**7**	8
⚂	4	5	6	**7**	8	9
⚃	5	6	**7**	8	9	10
⚄	6	**7**	8	9	10	11
⚅	**7**	8	9	10	11	12

There are 6 possible ways of making a sum of 7 with 2 dice, and so the probability of this happening is $\frac{6}{36}$ or $\frac{1}{6}$.

ii) A Double

Sum of throws	⚀	⚁	⚂	⚃	⚄	⚅
⚀	**2**	3	4	5	6	7
⚁	3	**4**	5	6	7	8
⚂	4	5	**6**	7	8	9
⚃	5	6	7	**8**	9	10
⚄	6	7	8	9	**10**	11
⚅	7	8	9	10	11	**12**

There are 6 possible ways of making a Double, so the probability of this happening is also $\frac{6}{36}$ or $\frac{1}{6}$.

iii) Sum of 6 or 8

Sum of throws						
	2	3	4	5	**6**	7
	3	4	5	**6**	7	**8**
	4	5	**6**	7	**8**	9
	5	**6**	7	**8**	9	10
	**6**	7	**8**	9	10	11
	7	**8**	9	10	11	12

There are 5 possible ways of making a sum of 6, and 5 ways of making 8. There are therefore 10 ways of making either score.

The probability is therefore $\frac{10}{36}$ or $\frac{5}{18}$.

iv) Double or sum of 7

Sum of throws						
	**2**	3	4	5	6	**7**
	3	**4**	5	6	**7**	8
	4	5	**6**	**7**	8	9
	5	6	**7**	**8**	9	10
	6	**7**	8	9	**10**	11
	**7**	8	9	10	11	**12**

There are 6 possible ways of making a Double, and 6 ways of making a 7. There are therefore 12 ways of making either. The probability is therefore $\frac{12}{36}$ or $\frac{1}{3}$.

Crucially (as we will see in the next case) these events are mutually exclusive – 7 is an odd number, and so you cannot have both a Double and a score of 7 with two dice.

v) Double or sum of 8

Sum of throws						
	**2**	3	4	5	6	7
	3	**4**	5	6	7	**8**
	4	5	**6**	7	**8**	9
	5	6	7	**8**	9	10
	6	7	**8**	9	**10**	11
	7	**8**	9	10	11	**12**

This might lead you to think that there are 11 possible ways of making either, but counting the highlighted squares only gives 10.

The probability is therefore $\frac{10}{36}$ or $\frac{5}{18}$.

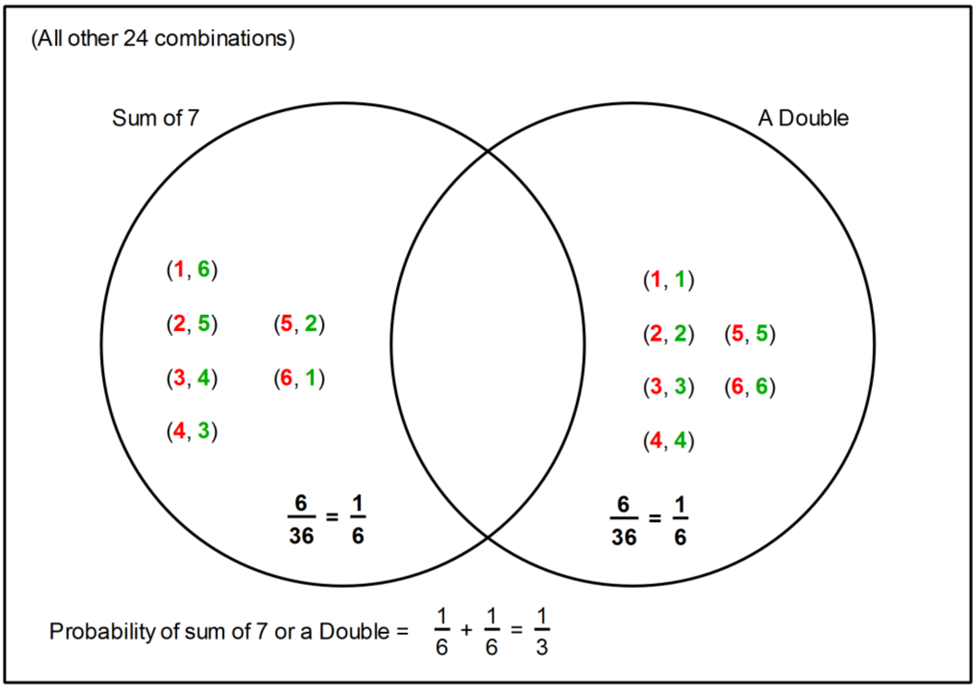
The reason for this discrepancy is that a Double and a sum of 8 are **not** mutually exclusive. Throwing a 4 with each dice gives both a Double and a sum of 8, so we have to take into account this probability, $\frac{1}{36}$, to prevent this event being counted twice.

In fact, P(Double or Sum of 8)= P(Double) + P(Sum of 8) – P(Double and Sum of 8).
$\rightarrow$ P(Double or Sum of 8) $= \frac{1}{6} + \frac{5}{18} - \frac{1}{36} = \frac{5}{18}$.

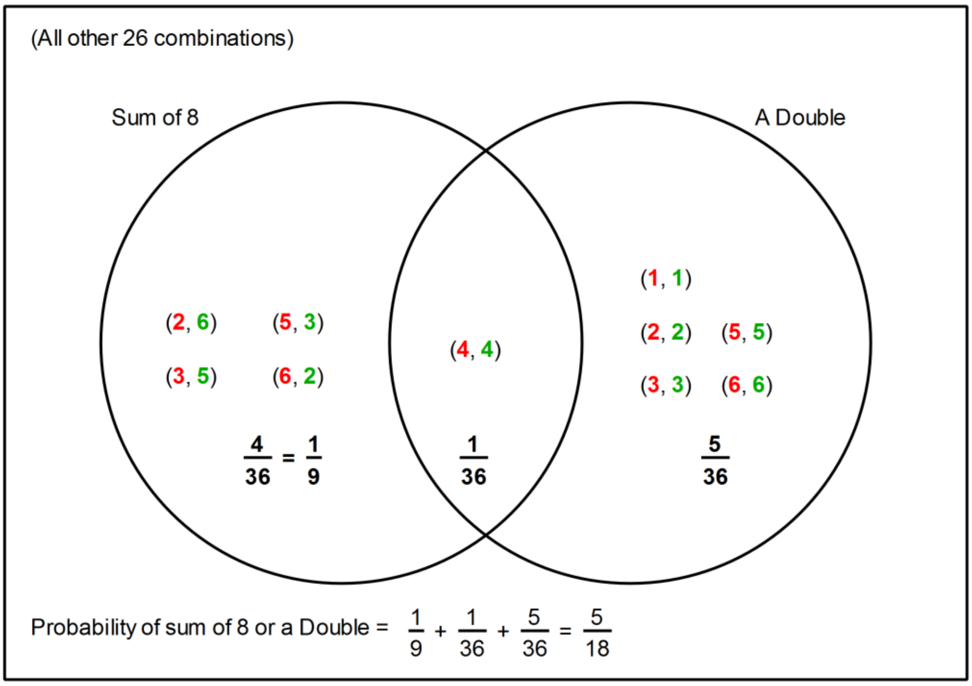
In general, if A and B are not mutually exclusive events, then the probability of either event occurring can be worked out by adding their individual probabilities together, then subtracting the probability of both occurring.

P(A or B) = P(A) + P(B) – P(A and B).

The probabilities in parts iv) and v) could also have been displayed on Venn diagrams:



It is impossible to throw a Double and a sum of 7 simultaneously, hence there are no entries in the overlapping region of the Venn diagram.



By contrast, it is possible to throw a Double and a sum of 8 if both dice land on 4, so this particular entry has to go in the overlapping region of the Venn diagram.

The Multiplication Rule for Independent Events.

Example (8):
A game involves a throwing a dice and drawing a playing card at random from a pack of 52.
What is the probability of obtaining (dice results quoted first):

i) an odd number and a spade?
ii) a 6 and a Jack ?
iii) a 1 and an Ace, or any number on the dice and a 'court card' ? (Court card is a King, Queen or Jack)

The first thing to notice here is the large number of different possible outcomes, actually 6 × 52 or 312. It would be hard work to draw up a possibility space diagram, so we make use of another rule.

If A and B are independent events, then the probability of both occurring is the product of the probabilities of each individual one occurring.

P(A and B) = P(A) × P(B).

This can be extended to three or more independent events.

i) The probability of throwing an odd number with the dice is $\frac{1}{2}$ (since half the numbers on a dice are odd) and that of drawing a spade is $\frac{1}{4}$ (since there are 4 suits with the same number of cards).
$\therefore$ The probability of an odd number and a spade is $\frac{1}{2} \times \frac{1}{4}$ or $\frac{1}{8}$.

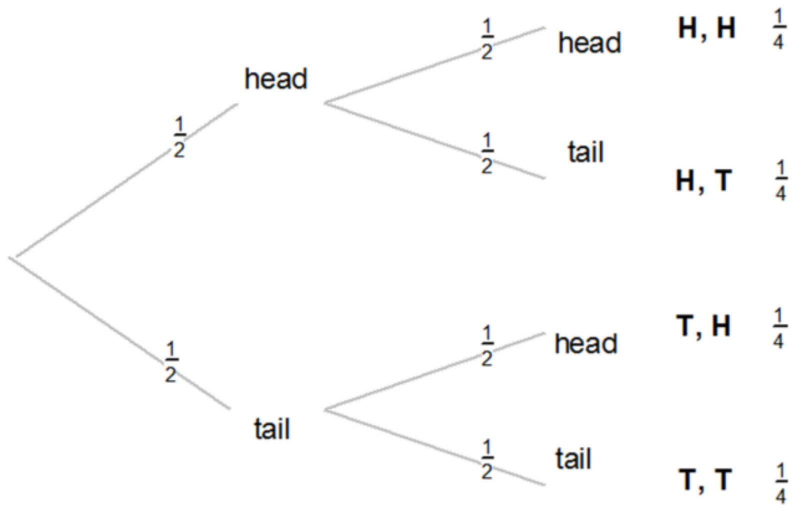
ii) The probability of throwing a 6 with the dice is $\frac{1}{6}$, and that of drawing a Jack is $\frac{1}{13}$ (since there are 4 Jacks in a pack of 52, or 1 in 13).
$\therefore$ The probability of a 6 and a Jack is $\frac{1}{6} \times \frac{1}{13}$ or $\frac{1}{78}$.

iii) The probability of scoring 1 on the dice and drawing an Ace follows the same rules as in part ii) , giving a value of $\frac{1}{78}$.
For the second part of the question, the result of the dice throw is of no importance, so the probability of 'any number' is just taken as 1. The probability of drawing a court card is $\frac{3}{13}$, as there are 12 such cards in a pack of 52.

$\therefore$ The probability of a 1 and an Ace, or any number and a 'court card' can be worked out by the addition rule for mutually exclusive events; here it is $\frac{1}{78} + \frac{3}{13}$, or $\frac{19}{78}$.

Probability Tree Diagrams.

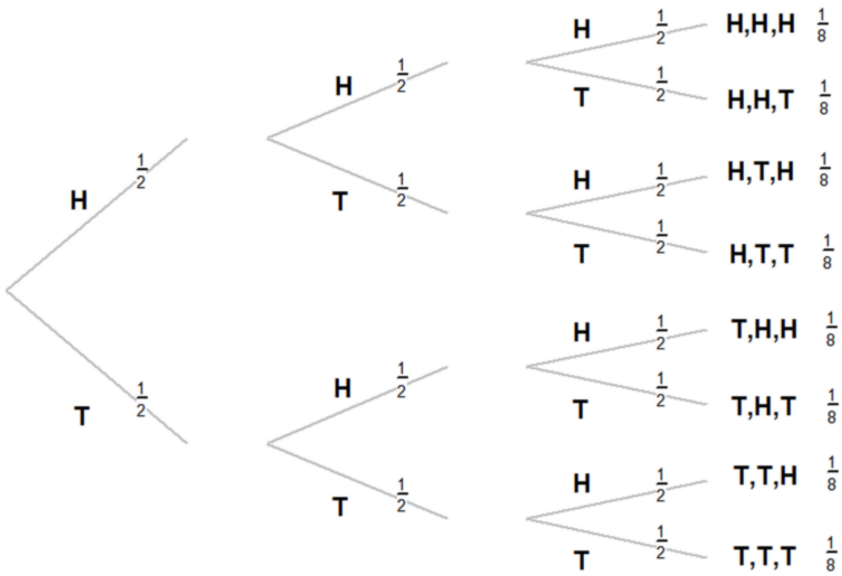
It is often convenient to use a tree diagram to work out probability problems, such as the one below showing the possible outcomes of tossing a fair coin twice.



This tree shows how the multiplication rule is used to calculate combined probabilities.
Note how there are two ways of obtaining a head and a tail, giving a total probability of $\frac{1}{2}$ for that event.

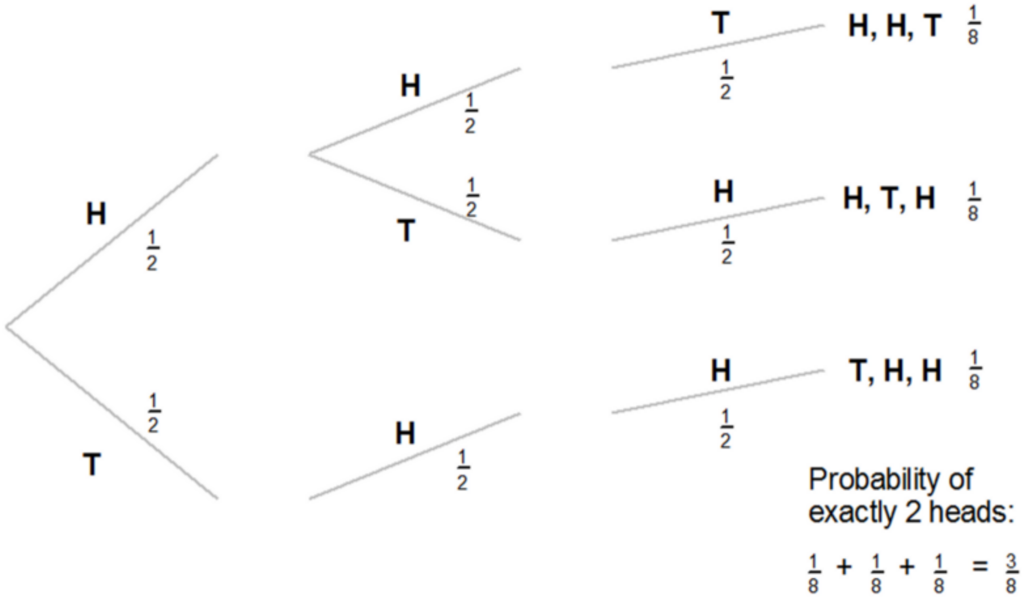
Example(9): Draw a probability tree diagram to show all the possible outcomes of tossing a coin three times.

This time, we have 8 distinct possibilities, each with equal probabilities of $\frac{1}{8}$.



It is not always necessary to draw a whole tree when solving probability problems, as the next example will show.

Example (10): Using a tree diagram, find the probability of tossing *exactly* two heads in three tosses of a fair coin.



Comparing the diagram with that for Example (8), we can see how the redundant branches of the tree have been 'pruned out', with only the required outcomes displayed on the right.

For example, if a 'tail' has been thrown on the first go, then both the following throws must be heads to satisfy the condition.

Thus it can be seen that the probability of exactly two heads in three tosses is $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}= \frac{3}{8}$.

Example (11): A marble is drawn from a bag containing 2 blue and 3 red marbles, its colour noted, and **the marble replaced** in the bag.

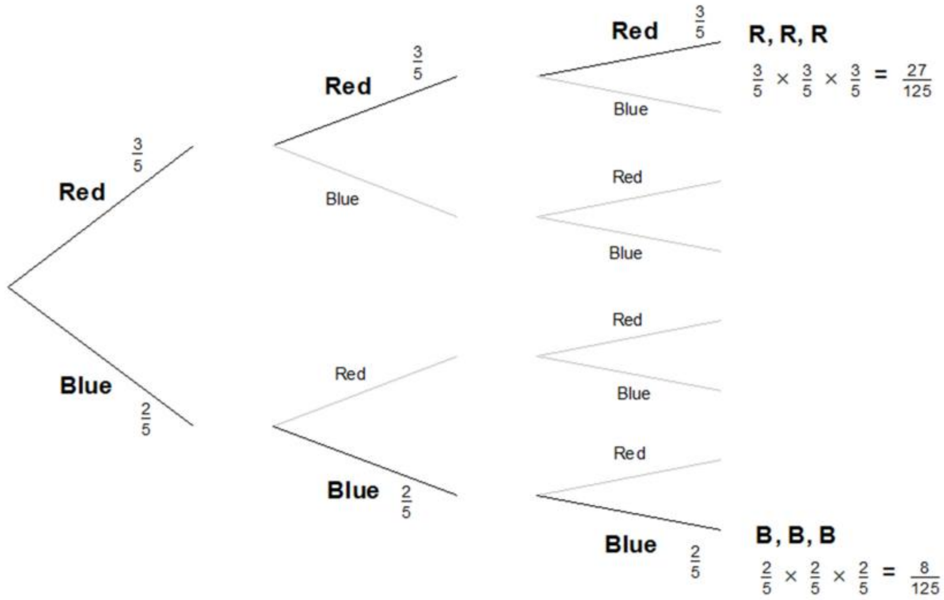
Find the probability that at least one blue and one red marble will be selected after 3 such draws.

The condition of "at least one blue and one red " can be interpreted to mean "do not include the case of three reds or three blues".

To refresh, there are eight possible outcomes: BBB, BBR, BRB, RBB, BRR, RBR, RRB and RRR.
It will be easier to find out the probabilities of three reds and three blues respectively, adding them, and subtracting from 1, a total of *two* outcomes, rather than trying to calculate the probabilities of *six* outcomes and adding them. The tree diagram illustrates this method.

Since there are 5 marbles in the bag in total, the probability of drawing a red each time is $\frac{3}{5}$ and that of drawing a blue is $\frac{2}{5}$. Because the marbles are drawn *with* replacement, the probabilities of a red and a blue do not differ between the first draw and the second.

(Redundant branches of the tree shown greyed out with smaller text).



Probability of "all reds" or "all blues" : $\frac{27}{125} + \frac{8}{125} = \frac{7}{25}$

Hence prob. of "at least one of each colour" : $1 - \frac{7}{25} = \frac{18}{25}$

Conditional probabilities.

So far, all the examples of tree diagrams referred to compound independent events, where the result of the first event had no bearing on the second.

Supposing we drew a card from a pack of 52 without putting it back in the pack. The probability of drawing an Ace is therefore $\frac{4}{52}$ or $\frac{1}{13}$.

The probability of drawing an Ace on the second draw will then depend on the result of the first one.

If the first card were not an Ace, there would still be 4 Aces left in the remaining 51 cards, giving a probability of $\frac{4}{51}$.

If the first card were an Ace, there would only be 3 Aces left in the remaining 51 cards, giving a probability of $\frac{3}{51}$ or $\frac{1}{17}$.

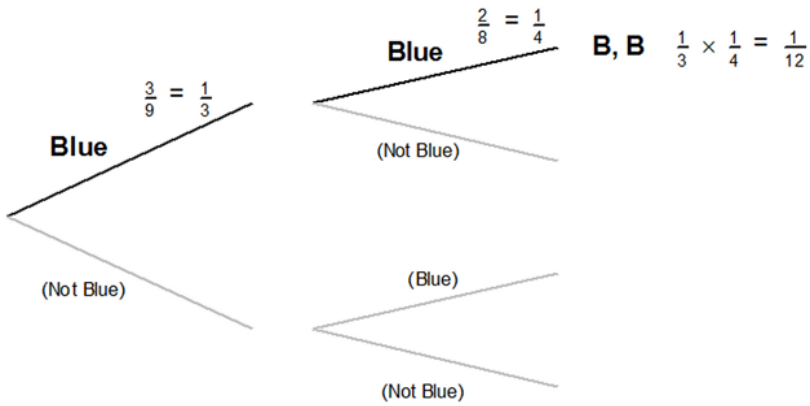
The example above demonstrates conditional probability, and this should be used whenever the question asks for **selection without replacement**.

Example (12): A marble is drawn from a bag containing 4 red, 3 blue and 2 green marbles, and then **not replaced** in the bag.

Find the following probabilities after two such draws, using tree diagrams:
i) both blue ii) exactly one red; iii) at least one green; iv) both the same colour.

Unlike the previous example, the marbles are *not* put back in the bag, and this alters the way in which the probabilities are calculated.

i) Both blue

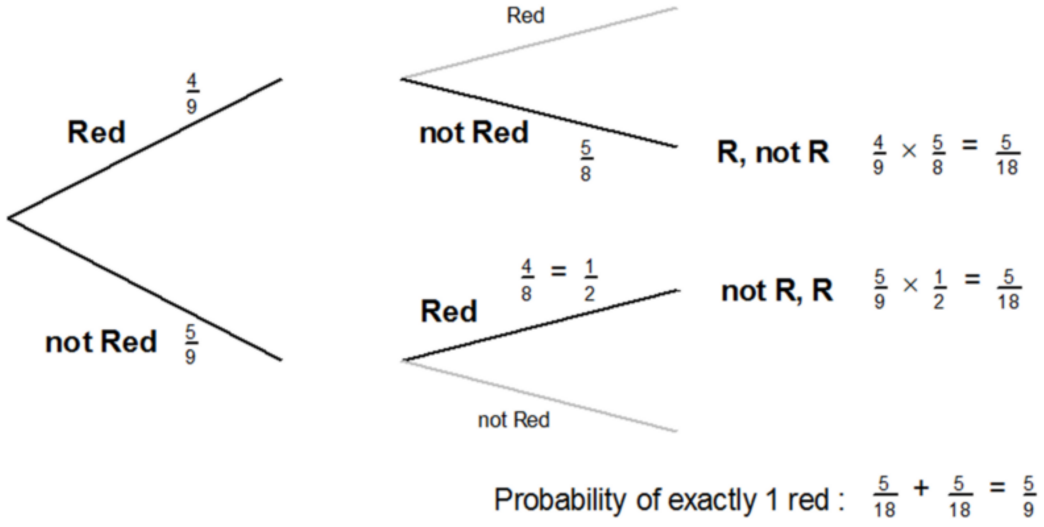


We are only interested in finding the probability of a blue, so we can combine the 'red' and 'green' draws into a 'not blue' category, which in any case is redundant.

Before the first draw, the probability of drawing a blue is $\frac{1}{3}$.
If the first marble drawn is blue, then there will be only 2 blue marbles left out of a total of 8 in the bag for the second draw, because there is no replacement.

Therefore, given that the first marble drawn is blue, the probability that the second one will be blue will be $\frac{1}{4}$, and hence by the product rule the probability that both will be blue is $\frac{1}{12}$.

ii) Exactly one red marble

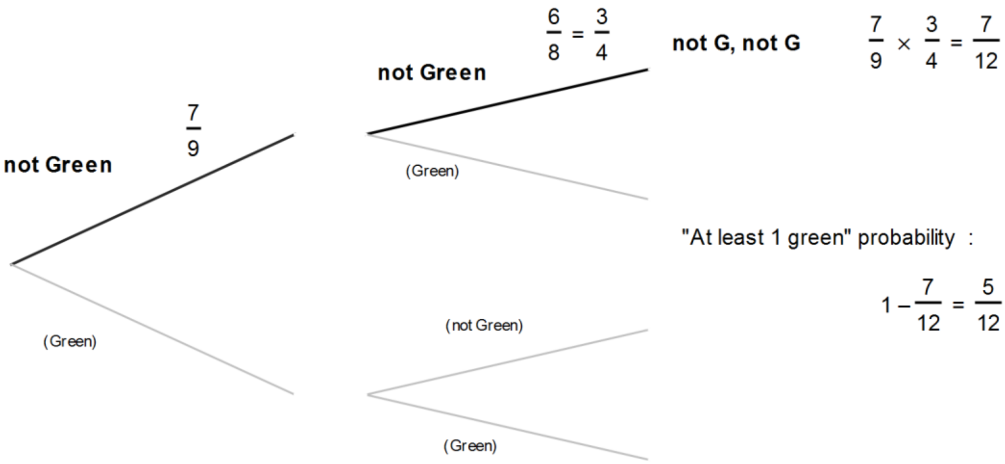


The only possible pair of draws satisfying the condition appears above.
We are not interested if the 'non-red' marble is blue or green, so we have combined the probabilities of 'blue' and 'green' into a 'not red' category.

If a red is drawn on the first draw, then 3 reds, and hence 5 'non-reds' will remain out of 8 in the bag, hence the probability of $\frac{5}{8}$ for a 'non-red' on the second draw.
If a 'non-red' is drawn on the first draw, then 4 reds will remain out of 8 in the bag, hence the probability of $\frac{1}{2}$ for a red on the second draw.

The probabilities of the two valid draws are then summed to give the overall probability of $\frac{5}{9}$.

iii) At least 1 green



The condition of "at least one green " can be interpreted to mean "do not include the cases where there are no greens at all".

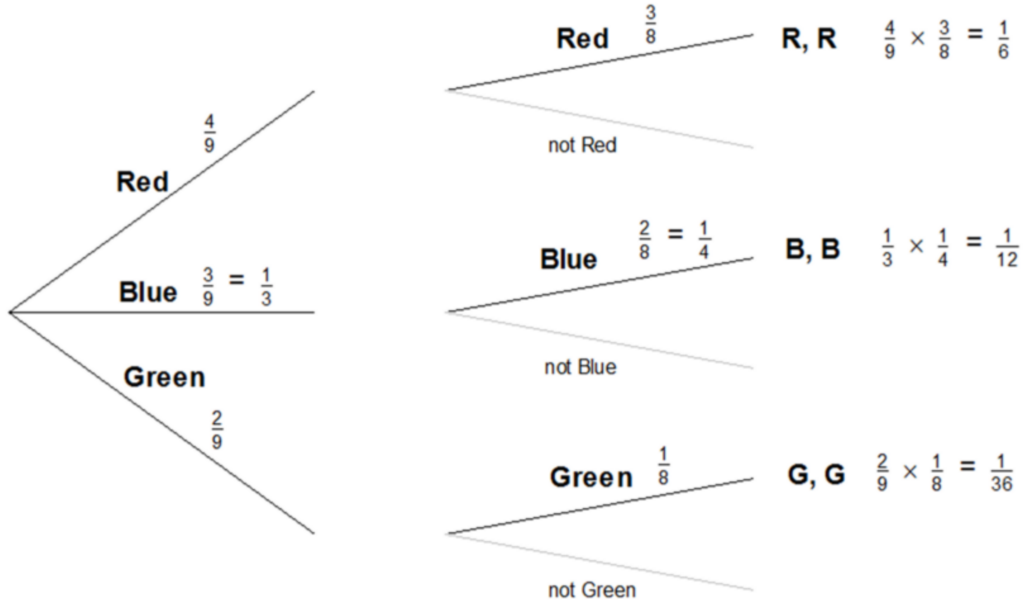
Again, we can lump the red and blue events into one category, 'not green'.

It will be easiest to find out the probability of 'not green' followed by another 'not green'..

The probability of 'not green' first time is that of green, namely $\frac{2}{9}$, subtracted from 1, hence the $\frac{7}{9}$.
If a 'not green' is drawn on the second draw, then 6 'not greens' will remain out of 8 in the bag, hence the probability of $\frac{3}{4}$ for a 'not green' on the second draw.

The probability of two 'not greens' works out as $\frac{7}{12}$, and so the probability of the opposite event, namely at least one green, works out as $1-\frac{7}{12}$ or $\frac{5}{12}$.

iv) Both marbles the same colour



Probability of both marbles the same colour: $\frac{1}{6} + \frac{1}{12} + \frac{1}{36} = \frac{5}{18}$

This time we need to have three branches to include each of the colours on the first draw, but again we can simplify the tree for the second draw, as we are not interested in the cases where the two marbles are of different colours.

Example 12(a): Ellie and Fiona have 90p and £1.60 in their purses respectively.
Neither girl has more than five coins in her purse, and the only coins present are 50p and 10p coins.

Ellie transfers one of her coins at random into Fiona's purse without Fiona seeing its type.
Fiona then transfers a coin at random from *her* purse back into Ellie's.

Calculate the probability that both girls' purses still have the starting sum of money after both transfers, in other words, 90p for Ellie and £1.60 for Fiona.

The only way that we can make 90p with 50p and 10p coins, and using fewer than five coins in total, is with **one 50p coin and four 10p coins**.

Similarly, the only way to make £1.60 with 50p and 10p coins under the same conditions is to use **three 50p coins and one 10p coin**.

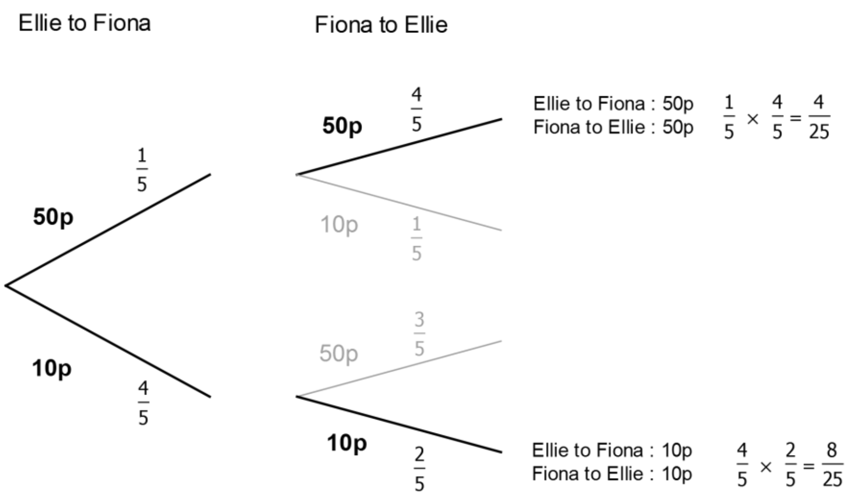
First transfer: Ellie to Fiona
Since Ellie has one 50p coin and four 10p coins in her purse at the start, the probability of her transferring a 50p coin into Fiona's purse is $\frac{1}{5}$, and transferring a 10p coin, it is $\frac{4}{5}$.

Second transfer : Fiona to Ellie
Fiona had three 50p coins and one 10p coin in her purse at the start, but then the ratios would have been altered after Ellie's transfer, with two results possible:

After Ellie transferring 50p to Fiona: Fiona has four 50p coins and one 10p coin
After Ellie transferring 10p to Fiona: Fiona has three 50p coins and two 10p coins

Each girl's purse must have its starting sum of money after the two transfers, and this can only occur if the two transferred coins are equal in value.

The full working is shown in the tree diagram below.



Prob. of same amount in each purse = $\frac{4}{25} + \frac{8}{25} = \frac{12}{25}$

More on Venn diagrams.

An alternative way of expressing probabilities of compound independent events is by using Venn diagrams, although this is limited to events with just two possible outcomes, such as heads / tails in a coin throw, or win / lose in a game where a draw is impossible.

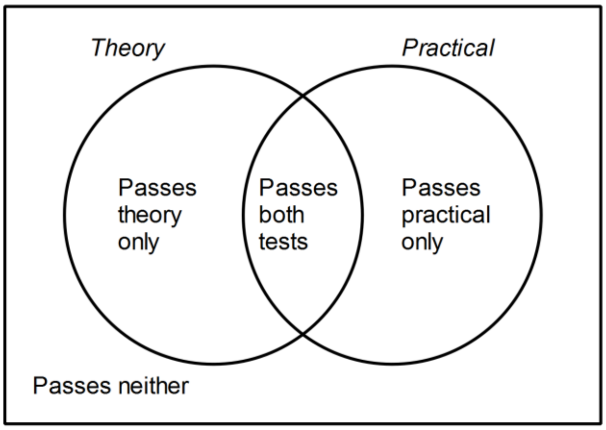
Example (13): Keith is taking his driving test, and has a 70% chance of passing his theory test, and an 80% chance of passing his practical test.

Show all of the possible outcomes, and their percentage probabilities, in a Venn diagram.

The events are shown as circles, where the inside of each circle represents a pass.
Since Keith can pass both tests, one test or none at all, there are four possible outcomes:

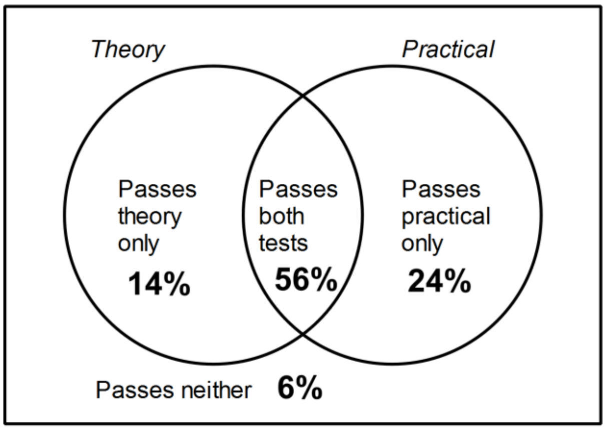
The region where the two circles overlap represents the case where Keith passes both tests.

The region in the box outside both circles represents the case of his failing both tests.



By the multiplication law, the probability of passing both the theory and practical tests is $0.7 \times 0.8 = 0.56$ (convert percentages to decimals !) or 56%

Since there are only two outcomes in each test, the probability of failing the theory test is $1 – 0.7$ or 0.3, and the probability of failing the practical test is 1- 0.8 or 0.2
Hence the probability of failing both tests is $0.3 \times 0.2 = 0.06$ or 6%.



This leave the cases where Keith passes only one test out of the two. These correspond to the 'outer' regions in each circle.

He has a 70% probability of passing the theory test, but we must subtract the case where he passes both, and so the probability of Keith passing the theory test alone is 70% - 56% = 14%.

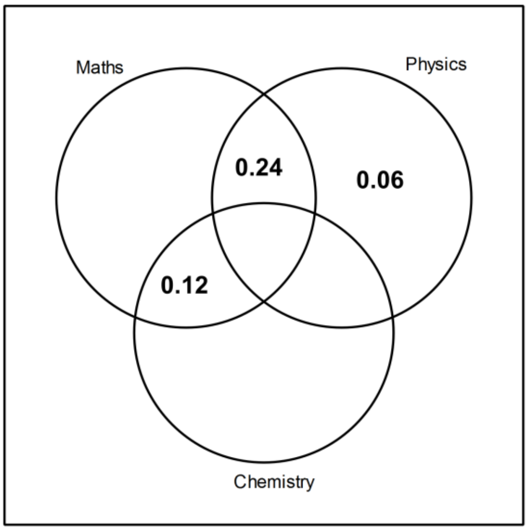
He has a 80% probability of passing the theory test, so again we must subtract 56%. The probability of Keith passing the practical test alone is 80% - 56% = 24%.

As a final check, all the probabilities add up to 56% + 6% + 14% + 24% = 100%., as they should !

Venn diagrams with three circles can also be used for combinations of three independent events.

Example (14): Lucy is taking her A-Level exams in Maths, Physics and Chemistry .

The probability of her receiving an 'A' grade is 0.8 for Maths, 0.75 for Physics and 0.6 for Chemistry.

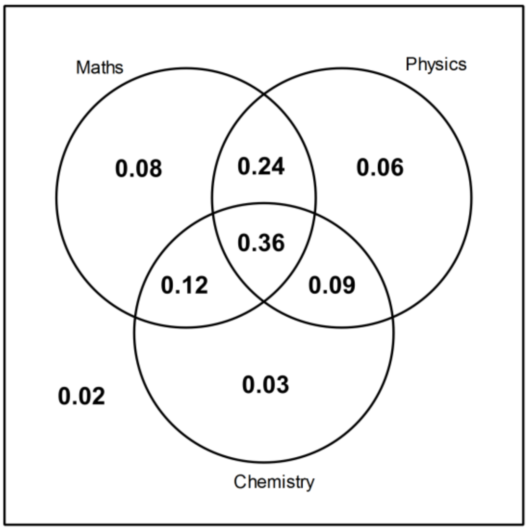


Complete this Venn diagram of probabilities of 'A' grades. Do not forget to include the situation where Lucy receives no 'A' grades.

The central region, where all three circles overlap, corresponds to Lucy receiving an 'A' grade in all three exams.

The probability of this event is
$0.8 \times 0.75 \times 0.6 = 0.36$ by the multiplication law.

After placing 0.36 in the central region, we can now calculate the values in all the other missing regions.



<u>'A ' grade in Maths only (upper left) :</u>
Since the total probability of an 'A' in Maths is 0.8, we just add the probabilities of the other three regions in the 'Maths' circle and subtract the result from 0.8. We enter $0.8 - (0.24 + 0.12 + 0.36) = 0.08$ there.

<u>'A ' grade in Physics and Chemistry, but not in Maths (centre right) :</u>
The total probability of an 'A' in Physics is 0.75, so we just add the probabilities of the other three regions in the 'Physics' circle and subtract that from 0.75. (Enter $0.75 - (0.24 + 0.06 + 0.36) = 0.09$ in there).

<u>'A ' grade in Chemistry only (lower part) :</u>
The total probability of an 'A' in Chemistry is 0.6, hence we just add the probabilities of the other regions in the 'Chemistry' circle and subtract that from 0.6. The result is $0.6 - (0.12 + 0.09 + 0.36) = 0.03$, so enter that .

This leaves us with the outer region, in the rectangle but outside all of the circles, corresponding to the case where Lucy gets no 'A' grades at all.

The probability of this event is $(1-0.8) \times (1-0.75) \times (1-0.6) = 0.2 \times 0.25 \times 0.4 = 0.02$.

A useful check is to add all the 8 probabilities together - they should all add up to 1 !

Frequency Trees.

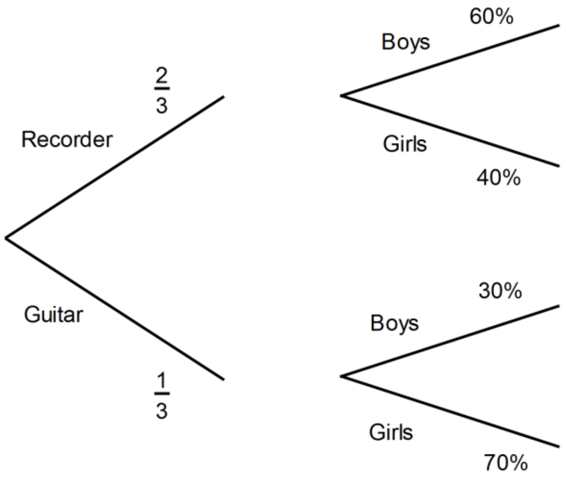
These diagrams are not unlike probability trees, but with extra numerical data added

Example (15): The pupils of Year 8 have a choice of learning to play either the recorder or the guitar should they wish to play a musical instrument.

In total, 60 pupils took up the challenge, and two-thirds chose to learn to play the recorder.
Of the pupils who chose the recorder, 60% of them were boys.
Of the pupils who chose the guitar, 70% were girls.

i) Complete the frequency tree.
ii) A girl pupil is chosen at random. What is the probability that she is learning to play the guitar ?

i) We can begin by setting up a probability tree as on the right.

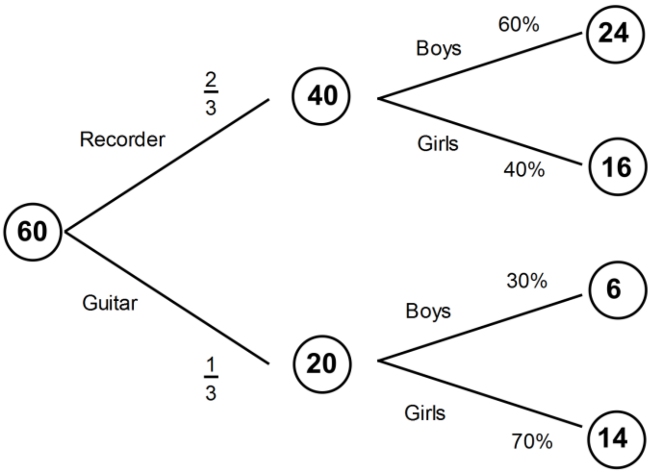


As we have 60 pupils in total, we can place the number 60 to the left of the leftmost branch.

Since two-thirds of 60 is 40, it follows that 40 pupils are learning the recorder and the remaining 20 are learning the guitar.

We therefore place 40 at the end of the "Recorder" branch and 20 at the end of the "Guitar" branch.

Now 60% of 40 is 24, so 24 boys are learning the recorder, as are 16 girls.



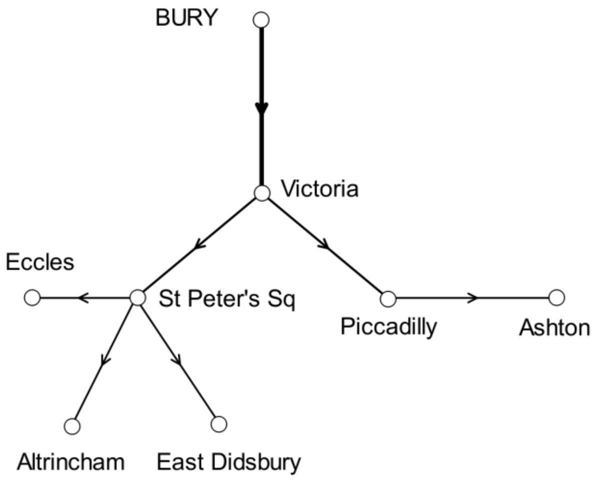
Similarly, 70% of 20 is 14, so 14 girls are learning the guitar, as are 6 boys.

ii) Since 14 girls are learning the guitar, and 16 the recorder, it follows that a girl chosen at random has a probability of $\frac{14}{14+16} = \frac{14}{30} = \frac{7}{15}$ that she is learning the guitar.

Frequency trees can also be used to analyse networks, as the next example shows.

Example (16): A future plan for Manchester's Metrolink tram system proposes that 24 trams will depart hourly from Bury each weekday and continue to Piccadilly, Ashton, Altrincham, East Didsbury and Eccles.

(This example does not claim to be an accurate representation of the Metrolink network, past, present, or future !)



All trams will continue to Victoria.
At Victoria, two-thirds of the trams will continue to St Peter's Square and the rest to Piccadilly.

Of the Piccadilly-bound trams, one quarter will terminate there and the remainder continue to Ashton.
Of the St Peter's Square – bound trams, half will continue to Altrincham, and the other half evenly split between continuing to Eccles or East Didsbury.

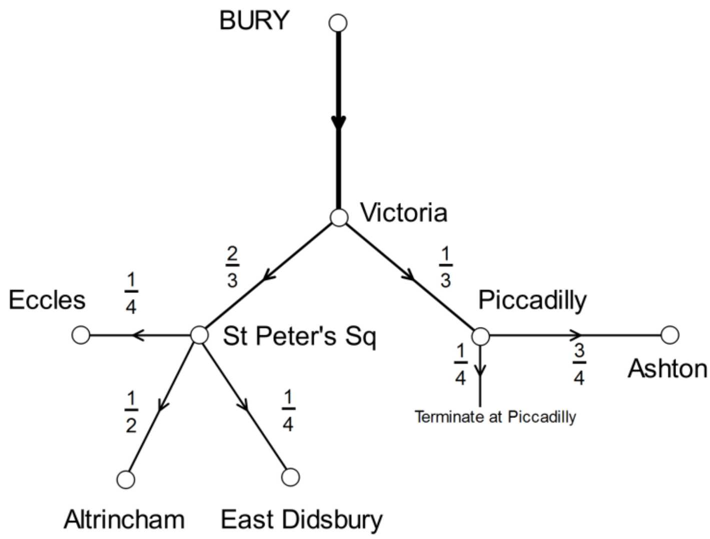
Complete the frequency tree, including all the fractions and hourly tram frequencies.
Hint: Include a "dummy" branch from Piccadilly to indicate that some trams terminate there.

The line from Bury branches off at Victoria, so we label the St Peter's Square and Piccadilly branches with the appropriate fractions, namely $\frac{2}{3}$ for the St Peter's Square branch and $1-\frac{2}{3}=\frac{1}{3}$ for the Piccadilly branch, remembering that the sum of all branches must be 1.

At Piccadilly, we add a "dummy" branch for the terminus and label it with the fraction $\frac{1}{4}$.

For the trams continuing to Ashton, we label that branch with $1-\frac{1}{4}=\frac{3}{4}$.

At St Peter's Square, half the trams continue to Altrincham and a quarter each to Eccles and East Didsbury, seeing as those two services are evenly split.
We therefore label the Altrincham branch with $\frac{1}{2}$ and the Eccles / East Didsbury branches with $\frac{1}{4}$ each, producing the probability tree on the right.



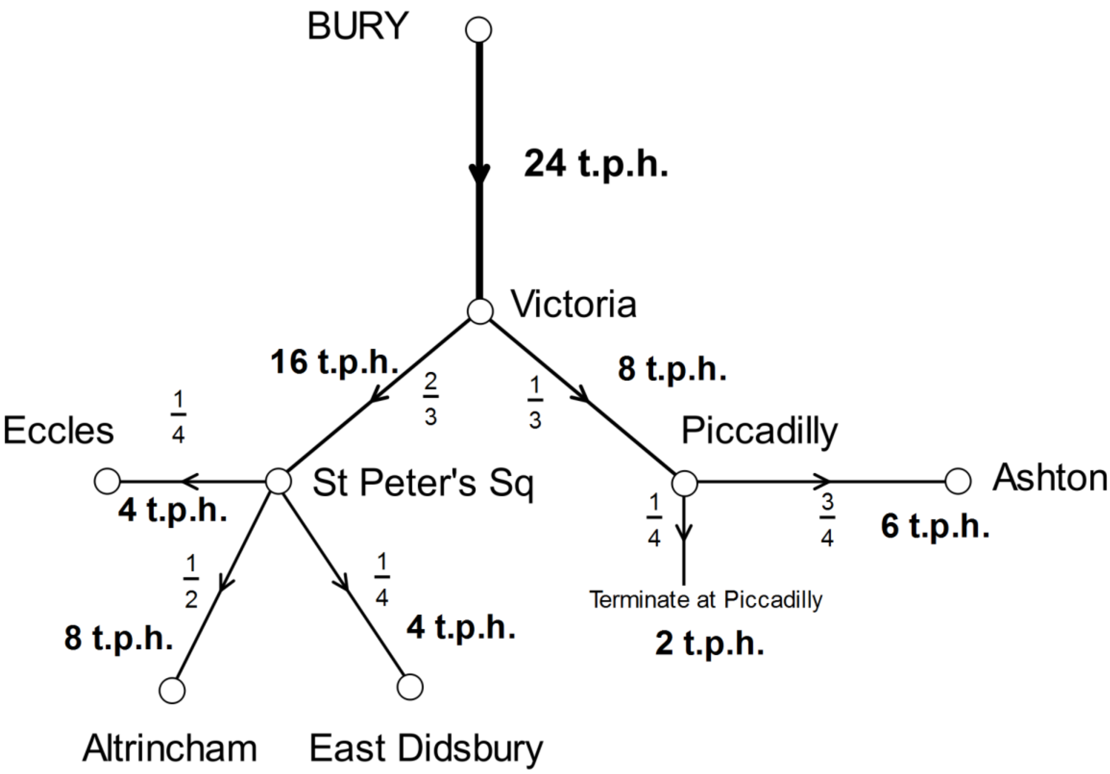
Finally, we change the probability tree into a frequency tree by calculating the frequencies of trams along the lines and branches.

We start with 24 trams leaving Bury each hour.

Two-thirds of 24 is 16, so 16 trams per hour continue to St Peter's Square and 8 to Piccadilly.

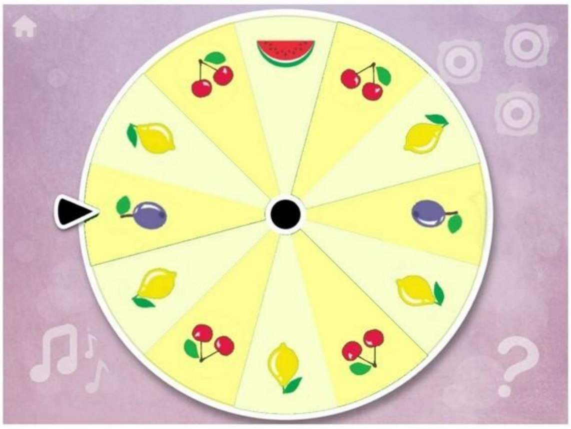
At Piccadilly, a quarter of those 8 hourly trams - that is 2 - terminate there, and the other 6 continue to Ashton.

Since 16 trams leave St Peter's Square every hour, half of them, or 8, continue to Altrincham, and the other 8 are split equally between East Didsbury and Eccles, which receive 4 trams per hour each.



Example (17a): Ben and Holly have designed a charity spinner game for their school garden party.

To play the game, a player must choose a fruit from a list, and then spin a wheel with 12 sectors. A win occurs if the wheel stops at the player's chosen fruit; in all other cases the player loses. The diagram below shows Ben's first attempt at setting up the game.



i) Kevin chooses the lemon and spins the wheel. Show that the probability of his winning is $\frac{5}{12}$.
Show that he would expect to lose £10 after 120 plays of the game, choosing the lemon each time.

ii) Holly then checks Ben's original figures and says, "I hope that punters don't go for the cherry too often, or for the plum. These might make the punter happy, but one of them actually loses for the school in the long term."

Show, using expected outcomes after 120 games, that Holly was right in her thinking.

i) Five of the 12 sectors of the wheel show a lemon, so Kevin has a $\frac{5}{12}$ probability of winning if he were to choose the lemon before the spin.

After 120 plays of the game, choosing the lemon each time, he would expect to win $\frac{5}{12}$ of 120, or 50, payouts of £1 each, for a total of £50. However, he would have paid 120 times 50p, or £60, in total, and so he would have lost £10, with his loss being the fund's gain.

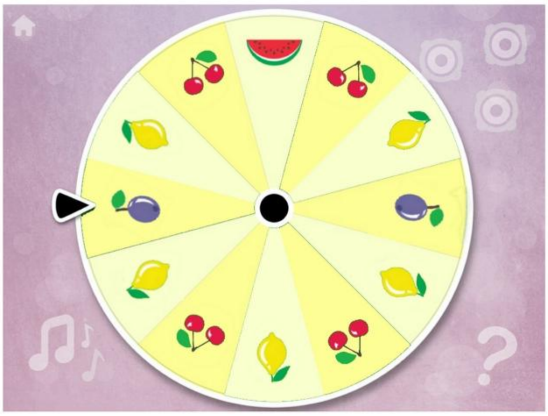
ii) If Kevin had chosen the plum instead of the lemon, and played 120 games, he again would have paid £60 into the game. He'd have a chance of $\frac{2}{12}$, or $\frac{1}{6}$, of making the right prediction because two out of the 12 sectors, or one-sixth, feature a plum. So he'd be expecting $\frac{1}{6}$ of 120, or 20, payouts of £3 each, also for a total of £60. The long-term result here is "no win, no lose", as the money paid out is expected to match the money paid in.

Should Kevin had chosen the cherry, and played 120 games as before, his chance of winning would be $\frac{4}{12}$, or $\frac{1}{3}$, since four of the 12 sectors feature a cherry.

He would be expected to win $\frac{1}{3}$ of 120, or 40 games, with a payout of £1.60 per game. Now 40 times £1.60 is £64, which would exceed the £60 he had paid in. So, Kevin would win in the long term, but the school would lose.

Therefore Holly was correct in her reasoning.

Example 17(b): Ben revised the prize money in the light of Holly's checks, resulting in the amended game layout :



Complete the table of expected payouts after playing 120 games, choosing each of the fruits in turn. Which choice is best for the player, and which one the best for the school fund ?

Chosen:	Probability	Expected frequency after 120 games	Paid into game	Paid out as winnings	Profit to school fund
Melon	$\frac{1}{12}$	10	£60	£4 × 10 = £40	£ 20
Plum			£60		
Cherry			£60		
Lemon			£60		

The probabilities are no different from those of the original game, and neither are the expected numbers of outcomes, so we can fill these details in straight away.

The winnings, however, need to be recalculated.

Chosen:	Probability	Expected frequency after 120 games	Paid into game	Paid out as winnings	Profit to school fund
Melon	$\frac{1}{12}$	10	£60	£4 × 10 = £40	£20
Plum	$\frac{2}{12}=\frac{1}{6}$	20	£60	£2.50 × 20 = £50	£10
Cherry	$\frac{4}{12}=\frac{1}{3}$	40	£60	£1.20 × 40 = £48	£12
Lemon	$\frac{5}{12}$	50	£60	£1 × 50 = £50	£10

As can be seen from the results table, the melon would be the worst choice for the player as the expected payout of £40 is only two-thirds of the £60 paid in, although it is best for the school fund as it brings in a profit of £20.

Choosing the plum or the lemon would on the other hand favour the player a little better, as the expected payout is £50, leaving a lower shortfall of £10 for the player, at the expense of a reduced profit for the school fund.

Example (18a): Lynda ran a spinner game at a charity fair last year, where contestants paid £2 per turn. The wheel had eight sectors of equal size as shown on the right.

According to Lynda, 400 punters played the game, with £102 raised for charity as a result.

Show, using full working, that this result corresponds to the expected outcome of 400 plays of the game.

If 400 people played the game, then Lynda would have received $400 \times £2$, or £800, from the players.

We need to calculate the expected payout after 400 plays of the game, hence we fill in the table as below.

The expected payout is obtained by multiplying the sum of money won per spin by the expected frequency.

(Since a "lost" result means no payout, we do not need to consider it here.)

Result of spin	**Relative frequency**	**Expected frequency after 400 spins**	**Expected Payout**
£3	$\frac{2}{8}=\frac{1}{4}$	100	£300
£4	$\frac{2}{8}=\frac{1}{4}$	100	£400

Hence, out of the £800 received, Lynda would have been expected to pay back £400 + £300, or £700, leaving a surplus of £100 for charity. This is in keeping with the actual sum raised of £102.

Example (18b): Lynda is planning to re-run the same game for charity this year, again priced at £2 per turn. This time she has re-designed the wheel so that each winning sector has had its angle reduced to 36°, but with the added possibility of winning £6 per game.

Calculate the expected money Lynda would raise for charity if, again, 400 people played the game.

Once again, Lynda would have received 400 × £2, or £800, from the players.

The difference between this example and the previous one is that each winning sector now only occupies 36°, or one tenth of a circle.

This reduces the probabilities of winning results, as the revised table below shows.

Result of spin	**Relative frequency**	**Expected frequency after 400 spins**	**Expected Payout**
£3	$\frac{2}{10} = \frac{1}{5}$	80	£240
£4	$\frac{1}{10}$	40	£160
£6	$\frac{1}{10}$	40	£240

The revised game looks at first to make little difference to the punter's chance of winning, by adding the possibility of a £6 win in spite of the reduced size of the winning sectors.

Examination of the payouts tells a different story. If Lynda were to receive £800 from 400 players, she would only be expected to pay back £240 + £160 + £240, or £640.

This gives an expected surplus of £800 - £640, or £160, after 400 plays of the game, compared to the £100 surplus of the last one . This game is better for the charity, but worse for the punters.

Many probability questions have an algebraic tie-in, such as this one based on a notorious examination question from a while back.

Example (19): There are n toffees in a bag, of which 6 are mint toffees and the rest treacle toffees. Ruth selects a toffee from the bag, eats it, and then selects a second one.

Given that the probability of Ruth selecting two mint toffees is $\frac{1}{8}$,

i) Show that $n^2 - n - 240 = 0$.

ii) Hence determine how many treacle toffees were in the bag at the start.

i) Since there are n toffees at the start, with 6 of them being mint toffees, the probability of Ruth selecting a mint toffee first time is $\frac{6}{n}$.

After Ruth has eaten one mint toffee, there are only 5 left out of a total of n-1. The probability of her selecting a second mint toffee is $\frac{5}{n-1}$, and hence the probability of her selecting two mint toffees is

$$\frac{6}{n} \times \frac{5}{n-1} = \frac{30}{n(n-1)}.$$

In addition we are given $\frac{30}{n(n-1)} = \frac{1}{8}$, rearranging to $n(n-1) = 240$, and finally to $n^2 - n - 240 = 0$.

ii) The equation readily factorises to $(n+15)(n-16) = 0$, giving solutions of $n = -15$ or $n = 16$.
Only the positive solution is valid here, so there are 16 toffees in the bag at the start, of which 6 are mint toffees and 16-6, or 10, are treacle toffees.