M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

A COMEDY OF ERRORS
(and how to avoid them)

Version: 2.4 Date: 03-04-2014

A COMEDY OF ERRORS

The main purpose of this section is to prevent unnecessary loss of marks in examinations.

Errors fall into various categories, namely

- Misreading the question
- Miscopying during the working
- Incorrect use of mathematics

Many of these errors can be avoided by reading the question carefully, managing time and checking over your work.

In each example, correct working is shown by a tick symbol and errors shown by a cross. An exclamation mark represents a possibility of some 'follow-through' credit despite an error in an earlier stage.

The written examples use the single arrow for implication, i.e.

$5x = 20 \rightarrow x = 4$ i.e. ' if $5x = 20$, then $x = 4$' , although the expression $5x = 20 \Rightarrow x = 4$ is the more correct form.

Errors caused by misreading the question.

Example (1): A TV set costs £220 before VAT at 20% is added to the price. How much is the VAT ?

Adding 20% is the same as multiplying by 1.20

$\therefore$ VAT = £220 × 1.20 = £264.

The pupil had calculated the price of the TV set plus the VAT : the question asked for the VAT only.
The correct working is :

VAT @ 20% on £220 = £(220 × 0.2) = £44

$\therefore$ the VAT is £44.

Example (2): Find the equation of a line perpendicular to $y = 1 - 2x$, passing through the point (2, 6).

Required line has gradient -2, so its equation is

$y = -2x + c$ or $y = c - 2x$.

Sub. $(x, y) = (2, 6)$, $c - 4 = 6$ $\therefore$ $c = 10$

so parallel line has equation $y = 10 - 2x$.

The pupil had misread the question and found the equation of the line *parallel* to $y = 1 - 2x$, not *perpendicular* to it. Correct is:

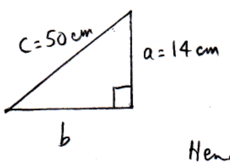
Gradient of $y = 1 - 2x$ is -2, so gradient of perpendicular line

$= \frac{1}{2}$, and its equation is $y = \frac{1}{2}x + c$

Sub. $(x, y) = (2, 6)$ $\rightarrow$ $1 + c = 6$ $\rightarrow$ $c = 5$

$\therefore$ eqn. of perpendicular line is $y = \frac{1}{2}x + 5$.

Example (3): A right-angled triangle has a hypotenuse of 50 cm and one short side of 14 cm. Find its perimeter.

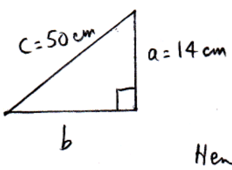


$b^2 = c^2 - a^2 = 50^2 - 14^2 = 2500 - 196 = 2304$

$\therefore b = 48$ cm

Hence area of triangle $= \frac{1}{2}(48 \times 14)$ cm^2 = 336 cm^2.

The question asked for the *perimeter* of the triangle, not the *area* !



$b^2 = c^2 - a^2 = 50^2 - 14^2 = 2500 - 196 = 2304$

$\therefore b = 48$ cm

Hence perimeter of triangle $= (50 + 14 + 48)$ cm $= 112$ cm.

Example (4): Find the median of the set of numbers 24, 14, 27, 21, 10, 26, 18.

Mean of 24, 14, 27, 21, 10, 26, 18 is $\frac{24+14+27+21+10+26+18}{7} = \frac{140}{7} = 20$

The pupil had calculated the *mean* instead of the *median*.

Sort in ascending order: 10, 14, 18, 21, 24, 26, 27

↑

Median = 21.

Example (5): Solve the equation $x^2 - 10x + 20 = 0$, giving your results to two decimal places.

$x^2 - 10x + 20 = 0 \rightarrow (x \cancel{-4})(x \cancel{-5}) = 0$

$-4 + -5 = -9$ so no good

$(x \cancel{-3})(x \cancel{-7}) = 0$

$(-3) \times (-7) = 21$, so no good

$(x - 2\frac{1}{2})(x - 7\frac{1}{2}) = 0$

$(-2\frac{1}{2}) \times (-7\frac{1}{2}) = 18\frac{3}{4}$, no good

? - - - - -

The pupil had wasted a lot of time and effort trying to factorise the quadratic expression. If the question mentions decimal places, or exact form, chances are the equation cannot be factorised. The equation should have been solved either by using the general formula or by completing the square, as in the correct answer below.

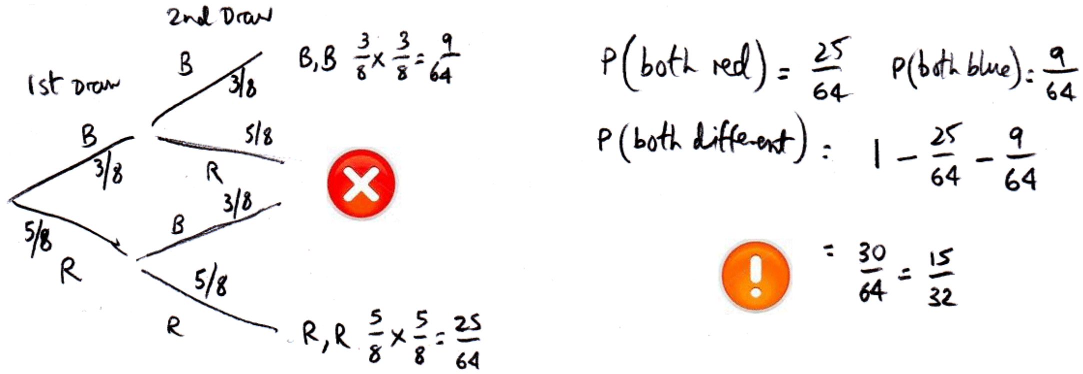
$x^2 - 10x + 20 = 0 \rightarrow$ where $a = 1$ $b = -10$ $c = 20$

sub into general formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{10 \pm \sqrt{100-80}}{2}$

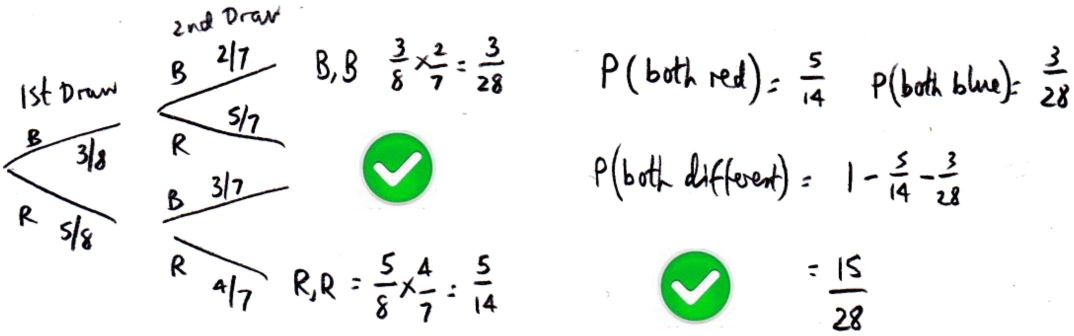
solutions $x = \frac{10 \pm \sqrt{20}}{2} \rightarrow x = 7.24$, $x = 2.76$ (2 dp).

Example (6): A bag contains 3 blue marbles and 5 red marbles. Laura draws one marble from the bag, and then a second marble without placing the first marble back in the bag.

Calculate the probability that i) both marbles are red; ii) the two marbles are of different colours.



The pupil had failed to realise that the first marble was not replaced, and that the probabilities were altered in the second draw. Correct is :



Errors caused by miscopying during the working.

Example (7): Evaluate $\sqrt{a^2+b^2+11}$ where a = 3 and b = 4.

$a = 3$
$b = 4$ $\rightarrow \sqrt{a^2+b^2+11} = \sqrt{3^2+4^2+11}$

$= \sqrt{25}+11 = 16.$

The pupil had blundered on the last step: the 11 was *inside* the square root sign up until then, but was incorrectly brought *outside* the square root sign, giving an incorrect final result. Be careful with expressions within square roots; if in doubt, use brackets to avoid confusion.

The correct working is shown below:

$a = 3$
$b = 4$ $\rightarrow \sqrt{a^2+b^2+11} = \sqrt{3^2+4^2+11}$

$= \sqrt{25+11} = \sqrt{36} = 6$

Example (8): Solve the equation $2x^2 - 3x - 1 = 0$, giving your answers correct to 2 decimal places.

$y = 2x^2 - 3x - 1 = 0 \rightarrow a, b, c = 2, -3, -1$

Roots are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\dfrac{3 \pm \sqrt{9+8}}{4} = \dfrac{3 \pm \sqrt{17}}{4}$

$= 3 + \dfrac{\sqrt{17}}{4},\ 3 - \dfrac{\sqrt{17}}{4}$ or $4.03,\ 1.97$ to 2 dp.

The pupil had substituted the correct values into the general quadratic formula, but the fraction $\dfrac{3 \pm \sqrt{17}}{4}$ was miscopied as $3 \pm \dfrac{\sqrt{17}}{4}$ because the fraction bar was not written clearly below the 3.

Correct working :

$y = 2x^2 - 3x - 1 = 0 \rightarrow a, b, c = 2, -3, -1$

Roots are $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{3 \pm \sqrt{9+8}}{4} = \dfrac{3 \pm \sqrt{17}}{4}$

$= \dfrac{3+\sqrt{17}}{4},\ \dfrac{3-\sqrt{17}}{4}$ or $1.78,\ -0.28$ to 2 dp.

Always write fractional expressions with the bar clearly shown !

Example (9): A line L_1 has the equation $y = 2x + 7$.

Line L_2 is perpendicular to L_1 and passes through the point (0, 1).

Find the coordinates of the point of intersection of the two lines.

Line $y = 2x + 7$ has a gradient of 2

∴ Perpendicular line has gradient of $-1/2$

Perpendicular line passes through (0, -1),

so its equation is $y = -\frac{1}{2}x - 1$

Lines intersect when $2x + 7 = -\frac{1}{2}x - 1 \rightarrow 2\frac{1}{2}x + 8 = 0$

$\rightarrow x = \frac{-16}{5}$. Sub. in $y = 2x+7$, $y = \frac{-32}{5} + 7 = \frac{3}{5}$

∴ Lines meet at $\left(\frac{-16}{5}, \frac{3}{5}\right)$.

The pupil had started well but miscopied the y-intercept of line L_2 as (0, -1) instead of (0, 1), carrying the error into the latter part of the question.

Correct answer:

Line $y = 2x + 7$ has a gradient of 2

∴ Perpendicular line has gradient of $-1/2$

Perpendicular line passes through (0, 1),

so its equation is $y = -\frac{1}{2}x + 1$ or $y = 1 - \frac{1}{2}x$

Lines intersect when $2x + 7 = 1 - \frac{1}{2}x \rightarrow 2\frac{1}{2}x + 6 = 0$

$\rightarrow x = \frac{-12}{5}$. Sub in $y = 2x+7$, $y = \frac{-24}{5} + 7 = \frac{11}{5}$

∴ Lines meet at $\left(\frac{-12}{5}, \frac{11}{5}\right)$.

Errors caused by incorrect mathematics.

The previous errors were all caused by carelessness in reading the question, or in miscopying at some stage of the working. This section deals with actual errors in the mathematics.

Dodgy Distributivity.

We know that $(a + b)\,x = ax + bx$ where x is any non-zero quantity, .e.g. $5 \times (3 + 4) = (5 \times 3) + (5 \times 4)$.

In other words, multiplication is distributive over addition.

Since division by a non-zero number is the same as multiplication by its reciprocal, the same rule holds, e.g. $\frac{3+2}{7} = \frac{3}{7} + \frac{2}{7}$

The same thing **cannot** be said of most other functions.

Example (10):
i) Work out $\sqrt{98} + \sqrt{2}$, simplifying your answer.
ii) Work out the value of sin 60°, given sin 30° = 0.5, and using a diagram if you wish.

(i) $\sqrt{98} + \sqrt{2} = \sqrt{100} = 10$

(ii) $\sin 60° = \sin(30° + 30°)$

$= \sin 30° + \sin 30° = 0.5 + 0.5 = 1$

In general,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b} \;;\; \sqrt{16+9} \neq \sqrt{16} + \sqrt{9}$$

$$\sin(a+b) \neq \sin(a) + \sin(b)\,;$$

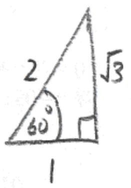
let $a = b = 45°$, $\sin(a + b) = \sin 90° = 1$, but $\sin 45° + \sin 45° = 1.41$ to 3 s.f.

The correct answers are shown below:

(i) $\sqrt{98} + \sqrt{2} = \sqrt{49}\sqrt{2} + \sqrt{2} =$

$= 7\sqrt{2} + \sqrt{2} = 8\sqrt{2}.$

(ii) $\sin 60° = \frac{\sqrt{3}}{2}$



Other examples of non-distributivity are:

$(a+b)^2 \neq a^2 + b^2$; proved by expanding, $(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 \neq a^2 - b^2$; proved by expanding, $(a-b)^2 = a^2 - 2ab + b^2$, and besides,

$a^2 - b^2 = (a+b)(a-b)$.

$\dfrac{1}{a+b} \neq \dfrac{1}{a} + \dfrac{1}{b}$; simplifying the RHS gives $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{b+a}{ab}$.

$x^{a+b} \neq x^a + x^b$; by the power laws, $x^{a+b} = x^a \times x^b$

Fractional Follies.

Example (11): Evaluate the following : i) : $\dfrac{1}{3} + \dfrac{2}{5}$; ii) $2\dfrac{2}{3} \times 3\dfrac{1}{2}$; iii) $\dfrac{6}{7} \div \dfrac{3}{4}$

$$\frac{1}{3} + \frac{2}{5} = \frac{1+2}{3+5} = \frac{3}{8}$$

$$\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{5+6}{15} = \frac{11}{15}$$

You cannot add (or subtract) fractions by simply adding (or subtracting) the top and bottom lines. You must find the L.C.M. of the denominators and work with equivalent fractions.

$$2\frac{2}{3} \times 3\frac{1}{2} = (2 \times 3) + \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} = 6 + \frac{1}{3} = 6\frac{1}{3}$$

$$2\frac{2}{3} \times 3\frac{1}{2} = \frac{\overset{4}{\cancel{8}}}{3} \times \frac{7}{\underset{1}{\cancel{2}}} = \frac{4 \times 7}{3} = \frac{28}{3} = 9\frac{1}{3}$$

Never try and multiply mixed numbers without changing them into improper fractions first. You certainly cannot multiply the whole numbers and fractions separately and then add the results.

$$\frac{6}{7} \div \frac{3}{4} = \frac{7}{\underset{2}{\cancel{6}}} \times \frac{\overset{1}{\cancel{3}}}{4} = \frac{7}{2} \times \frac{1}{4} = \frac{7}{8}$$

$$\frac{6}{7} \div \frac{3}{4} = \frac{\overset{2}{\cancel{6}}}{7} \times \frac{4}{\cancel{3}} = \frac{8}{7} = 1\frac{1}{7}$$

When we divide fractions, we invert the fraction to the *right* of the division sign, not the left !

Example (12): Convert i) the recurring decimal $0.0\dot{6}$ into an ordinary fraction in its lowest terms; ii) the fraction $\frac{7}{12}$ into a recurring decimal.

i)

$$F = 0.0\dot{6} = 0.0606\ldots$$
$$100F = 6.06\ldots = 6.0606\ldots$$
$$99F = 6.000$$
$$\therefore F = \frac{6}{99} = \frac{2}{33}$$

$$F = 0.0\dot{6} = 0.0666\ldots$$
$$10F = 0.\dot{6} = 0.6666\ldots$$
$$9F = 0.6 = 0.6$$
$$F = \frac{0.6}{9} = \frac{6}{90} = \frac{1}{15}.$$

The incorrect result on the left was a result of confusing $0.0\dot{6}$ (where only the 6 recurs, not the 0) with $0.\dot{0}\dot{6}$ (where the pair of digits 06 recurs). The correct working result is on the right.

ii)

$$12 \overline{)\,7.000\,}\quad 0.583 \qquad \frac{7}{12} = 0.583$$

$$12 \overline{)\,7.0000\,}\quad 0.5833 \qquad \frac{7}{12} = 0.58\dot{3}$$

remainder recurs

The partly correct result on the left was due to the pupil not continuing with the division until the recurring nature of the decimal became evident.

Example (13): Convert: i) $\frac{1}{3}$ into a percentage; ii) 0.7% into a decimal fraction.

$$\frac{1}{3} = 33\%$$
$$0.7\% = \frac{7}{100} = 0.07$$

$$\frac{1}{3} = \left(100 \times \frac{1}{3}\right)\% = \frac{100}{3}\% = 33\tfrac{1}{3}\%$$
$$0.7\% = \frac{0.7}{100} = \frac{7}{1000} = 0.007$$

The incorrect result for i) was a rounding issue, as 3 times 33% is 99%, not 100%. In the incorrect result for part ii) the pupil mistook 0.7% for 7%.

Example (14): Express the fraction $\frac{26}{65}$ in its lowest terms.

$$\frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}$$

This is a ridiculous, but amusing, instance of the pupil obtaining the correct result, but using a totally incorrect method. The 6’s were ‘cancelled out’ as if they were factors !

As a matter of interest, $\frac{16}{64}$ and $\frac{19}{95}$ can also be reduced to their lowest terms using this “method”.

The correct method is to factorise the top and bottom lines.

$$\frac{26}{65} = \frac{2\times\cancel{13}}{5\times\cancel{13}} = \frac{2}{5}$$

Perilous Percentages.

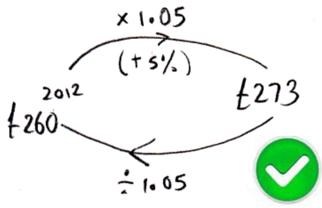
Example (15): Fred’s annual home insurance cost him £273 in 2013, or an increase of 5% over 2012. What did Fred pay for his home insurance in 2012 ?

Insurance costs £273 in 2013

$\therefore$ increase $= \frac{5}{100} \times 273 = £13.65$
5%

$\therefore$ Fred paid £273 − £13.65 in 2012, or £259.35.

The pupil had failed to see that this was a *reversed* percentage question, where a value *after* the % change was given, and the value *before* the change had to be found.



Fred's insurance in 2012 was

$$£\frac{273}{1.05} = £260.$$

Note that, although multiplying by 1.05 is the same as adding 5%, dividing by 1.05 is *not* the same as subtracting 5%.

Example (16): George sees his shares decrease in value by 20% one day and increase by 25% the following day. He reckons that he had made a gain of 5%. Is he correct ?

$100\% - 20\% + 25\% = 105\%$ ∴ George made a 5% gain

The pupil simply added the percentage changes to come up with a net gain of 5%, but this is wrong because whilst the decrease of 20% applies to the total original value, the increase of 25% applies to only 80% of the total original, not 100% of it.

George's investment starts (say) at £100.
20% loss gives £100 × 0.8 or £80.
25% gain on £80 gives £80 × 1.25 or £100.
∴ George's shares haven't changed in value at all.

Example (17): Jane invests £4000 over 4 years in an account paying 3.5% compound interest per annum. How much will she have in her account at the end of the 4 years ?

Jane invests for 4 years, principal = £4000, rate = 3.5%
∴ Interest earned = $£\left(4000 \times 4 \times \frac{3.5}{100}\right) = £560$
Jane will have £4560 in the account after 4 years.

The pupil had worked out the *simple* interest over 4 years instead of the *compound* interest.
A correct (but long-winded) method is shown below.

Interest in Yr. 1 : 3.5% of £4000 = £140
— " — 2 : 3.5% of £4140 = £144.90
3 3.5% of £4284.90 : £149.97
4 3.5% of £4434.87 = £155.22
Total Interest £590.09

∴ Account after 4 years = £4590.09

All that is needed to work out the value of the investment is this : .

Account after 4 years $= £(4000 \times 1.035^4) = £4590.09$

Adding 3.5% to a quantity is the same as multiplying it by 1.035.

Infuriating Indices.

Example (18): Write down the values of i) 5^0 ; ii) 2^{-3}.

$$\text{(i) } 5^0 = 0 \text{ ; (ii) } 2^{-3} = -2^3 = -8$$

Those are two big blunders; the zero power of any positive number is always 1, not 0; also the negative power of a number is the *reciprocal* of the corresponding positive power and not the *negative* of it !

$$\text{(i) } 5^0 = 1 \quad \text{(ii) } 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Example (19): Find the value of i) $16^{½}$; ii) $4^2 \times 2^3$

$$\text{(i) } 16^{1/2} = 8 \text{ ; } \quad \text{ii) } 4^2 \times 2^3 = 8^5$$

A number raised to the power of ½ is the square root of the number; also, when performing index arithmetic, we cannot mix up powers of different numbers.

$$\text{(i) } 16^{1/2} = \sqrt{16} = 4 \text{ ; } \quad \text{ii) } 4^2 \times 2^3 = (2^2)^2 \times 2^3$$

$$= 2^4 \times 2^3 = 2^7.$$

Notice how we had to redefine 4 as the square of 2 to continue with part ii).

Surd Absurdities.

We've seen surds mishandled in Example (10): here are a few more.

Example (20): Express the following as square roots of a single number;

i) $\sqrt{2} \times \sqrt{7}$; ii) $3\sqrt{5}$; iii) $\dfrac{\sqrt{28}}{\sqrt{2}}$; iv) $\dfrac{\sqrt{20}}{2}$

(i) $\sqrt{2} \times \sqrt{7} = \sqrt{2 \times 7} = \sqrt{14}$ ii) $3\sqrt{5} = \sqrt{3 \times 5} = \sqrt{15}$

iii) $\dfrac{\sqrt{28}}{\sqrt{2}} = \sqrt{\dfrac{28}{2}} = \sqrt{14}$ iv) $\dfrac{\sqrt{20}}{2} = \sqrt{\dfrac{20}{2}} = \sqrt{10}$

The pupil had failed to express 3 as √9 in part ii) and 2 as √4 in part iv).

(i) $\sqrt{2} \times \sqrt{7} = \sqrt{2 \times 7} = \sqrt{14}$ (ii) $3\sqrt{5} = \sqrt{9}\sqrt{5} = \sqrt{9 \times 5} = \sqrt{45}$

iii) $\dfrac{\sqrt{28}}{\sqrt{2}} = \sqrt{\dfrac{28}{2}} = \sqrt{14}$ (iv) $\dfrac{\sqrt{20}}{2} = \dfrac{\sqrt{20}}{\sqrt{4}} = \sqrt{\dfrac{20}{4}} = \sqrt{5}$

Example (21): Rationalise the denominator in the expressions i) $\dfrac{1}{\sqrt{3}}$ ii) $\dfrac{4}{3+\sqrt{7}}$

(i) $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

(ii) $\dfrac{4}{3+\sqrt{7}} = \dfrac{4}{3+\sqrt{7}} \times \dfrac{3+\sqrt{7}}{3+\sqrt{7}} = \dfrac{12+4\sqrt{7}}{9+7} = \dfrac{12+4\sqrt{7}}{16} = \dfrac{3+\sqrt{7}}{4}$

In ii), the pupil incorrectly expanded the square of (3 + √7), thinking it gave a rational result. The required multiplier to rationalise the denominator is (3 - √7), as shown below.

(i) $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

(ii) $\dfrac{4}{3+\sqrt{7}} = \dfrac{4}{3+\sqrt{7}} \times \dfrac{3-\sqrt{7}}{3-\sqrt{7}} = \dfrac{12-4\sqrt{7}}{9-7} = \dfrac{12-4\sqrt{7}}{2} = 6-2\sqrt{7}$

More Number Nastiness.

Example(22) : Give the values of: i) 2.4183 to 2 decimal places; ii) 5.318 to 2 significant figures; iii) 4207.6 to 3 significant figures.

(i) 2.4183 = 2.41 (2 dp) (ii) 5.318 = 5.3 (2 s.f.)
(iii) 4207.6 = 421 (3 sf)

In part i), the pupil forgot to round up the second decimal place. In part iii) the pupil forgot to realise that the result was ten times smaller than it should have been, i.e. the place values of the significant digits were wrong.

(i) 2.4183 = 2.42 (2 dp) (ii) 5.318 = 5.3 (2 s.f.)
(iii) 4207.6 = 4210 (3 sf)

In part iii) we had to place a zero in the units digit to keep the place values correct .

Example (23) A builder's lorry can carry a maximum safe load of 25 tonnes to the nearest tonne and delivers pallets of sand to a building site. If a pallet of sand weighs 750 kg to the nearest 50 kg, what is the maximum number of pallets that can be loaded safely on to the lorry ?

Upper bound of permissible load : 25,500 kg
Upper bound of pallet weight : 775 kg

$\therefore$ Maximum safe no. of pallets : $\frac{25500}{775} = 32.90..$

= 32 to nearest integer below.

It is fairly obvious that multiplying the upper bounds of two numbers gives a maximum possible result, but such a generalisation does not hold true in division.

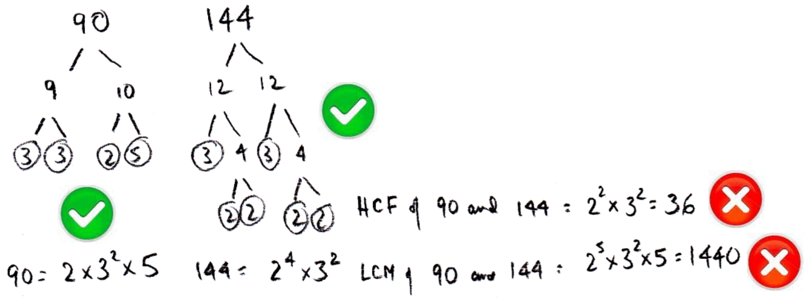
The worst-case scenario in this case is when we divide the *lower* bound of the load by the *upper* bound of the pallet weight.

Lower bound of permissible load : 24,500 kg
Upper bound of pallet weight : 775 kg

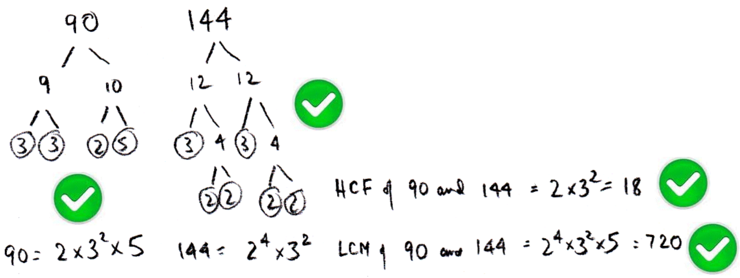
$\therefore$ Maximum safe no. of pallets : $\frac{24500}{775} = 31.61..$

= 31 to nearest integer below

Example (24) : i) Separate 90 and 144 into their prime factors. ii) Hence find the H.C.F. and L.C.M. of 90 and 144.



The numbers were separated correctly into their prime factors, but the pupil had miswritten the power of 2 in the H.C.F. and unnecessarily multiplied the powers of 2 in the L.C.M.



The number 2 occurs in 90 only as 2 itself ; in 144, it occurs as 2^4.
The *lower* power, i.e. 1, is in the H.C.F. ; the *higher* power, i.e. 4, is in the L.C.M.

Example (25): The instructions on the label of a 2.5 litre bottle of fruit squash say "Dilute to taste: 1 part squash to 5 parts water". How many cups of capacity 250 ml can be filled from the contents of this bottle after carrying out the recommended dilution ?

Ratio Squash : Water = 1:5
After diluting, vol. of squash
= 2.5 × 5 = 12.5 litres ∴ $\frac{12500}{250}$, or 50, cups can be filled

The pupil calculated the amount of *water only* in the diluted squash, instead of combining the volumes of the water and the undiluted squash. Correct working is shown below

Ratio Squash : Water = 1:5
1 + 5 = 6, so 1 part undiluted
squash yields 6 parts diluted.
Vol. after diluting = 2.5 × 6 = 15 litres ∴ $\frac{15,000}{250}$, or 60, cups can be filled.

Example (26):
i) The mean distance of the Moon from the Earth is 384,300 km. Express this distance (in km) in standard form.
ii) The wavelength of red light is 6.56×10^{-7} m. Express this as an ordinary number.
iii) Find the value of $(3.5 \times 10^{6}) \times (4 \times 10^{3})$.

(i) $384{,}300 \text{ km} = 3.843 \times 10^{6} \text{ km}$

(ii) $6.56 \times 10^{-7} \text{ m} = 0.0000000656 \text{ m}$

(iii) $(3.5 \times 10^{8}) \times (4 \times 10^{3}) = 14 \times 10^{11}$.

In i) the pupil thought that, since the number 384300 had 6 digits, the power of 10 would be 6.
In fact it is 5, as $100{,}000 = 10^{5}$ and $1{,}000{,}000 = 10^{6}$.

In ii) the pupil inserted 7 zeros after the decimal point instead of just 6.
Take $10^{-3} = 0.001$ (2 zeros after decimal point, not 3)

In iii) the end result was 'mongrel' standard form because the number part was not between 1 and 9.9999... . The pupil should have divided the 14 by 10 and added 1 to the power to compensate.

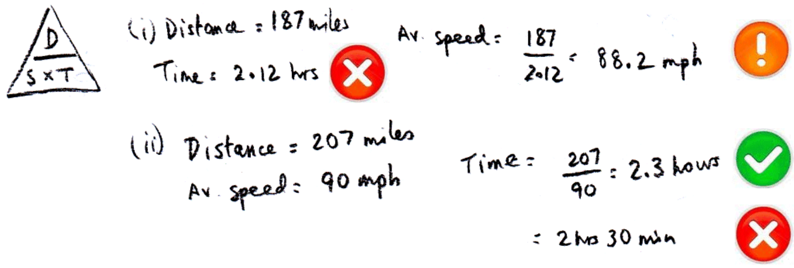
Correct results below:

(i) $384{,}300 \text{ km} = 3.843 \times 10^{5} \text{ km}$

(ii) $6.56 \times 10^{-7} \text{ m} = 0.000000656 \text{ m}$

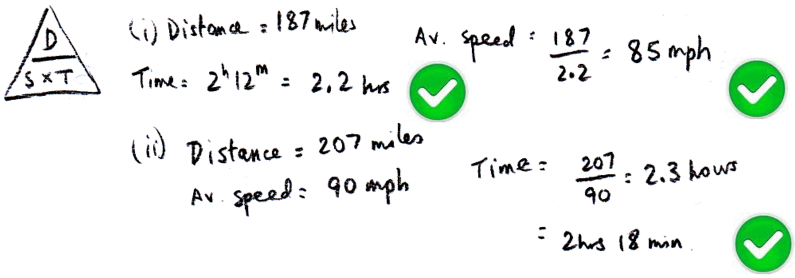
(iii) $(3.5 \times 10^{8}) \times (4 \times 10^{3}) = 14 \times 10^{11} = 1.4 \times 10^{12}$.

Example (27): i) A train journey takes 2 hours and 12 minutes from Manchester Piccadilly to London Euston, a distance of 187 miles. What is the average speed of the train in mph ?
ii) Another train covers the 207 miles from London Euston to Preston at an average speed of 90 mph. What is the travel time ?



Units of time are not wholly decimal !

In i) 2 hours and 12 minutes is equal to 2.2 hours in decimal form, not 2.12 hours. Similarly in ii) 2.3 hours is equal to 2 hours and 18 minutes, not 2 hours and 30 minutes.



Algebraic Abuse.

Example (28): Expand i) $x(5x - 3)$; ii) $(x + 5)^2$; iii) $-2a(3a - 5)$ and iv) factorise $6x^2y + 10x^3y^2$ fully.

(i) $x(5x+3) = 5x^2+3$

(ii) $(x+5)^2 = x^2+25$

(iii) $\overbrace{\underbrace{-2a(3a-5}_{-10a}}^{-6a^2}) = -6a^2 - 10a$

(iv) $6x^2y + 10x^3y^2 = xy(6x + 10x^2y)$

In i) , the pupil forgot to multiply the second term inside the brackets.
In ii), the pupil thought wrongly that $(a+b)^2 = a^2 + b^2$ for all a , b; the correct form being $(a+b)^2 = a^2 + 2ab + b^2$

In iii), the pupil forgot to reverse the sign of the second term when multiplying by $-2a$.
In iv), the pupil failed to spot that $6x$ and $10x^2y$ have a common factor of $2x$, which should have been taken outside the brackets.

Corrected answers :

(i) $x(5x+3) = 5x^2 + 3x$

(ii) $(x+5)^2 = (x+5)(x+5) = x^2 + 10x + 25$

x^2 25 $+5x$ $+5x$

(iii) $\overbrace{\underbrace{-2a(3a-5}_{+10a}}^{-6a^2}) = 10a - 6a^2$

(iv) $6x^2y + 10x^3y^2 = 2x^2y(3 + 5xy)$

Example (29) (intro): Here is a rather strange fraction sum, reminiscent of Example 14:

$$\frac{\overset{4}{\cancel{40}}+16}{\underset{1}{\cancel{10}}+4} = \frac{4+16}{1+4} = \frac{20}{5} = 4 \qquad \frac{\overset{4}{\cancel{40}}+14}{\underset{1}{\cancel{10}}+8} = \frac{4+14}{1+8} = \frac{18}{9} = 2$$

The starting expression in the first case is equal to $\frac{56}{14}$. We then tried cancelling a factor of 10 from one of the terms on the top and bottom, and ended up with a result of $\frac{20}{5}$. Both results happened to be equal to 4, but this method does not work in general.

In the second case, $\frac{54}{18}$, or 3, is certainly not equal to $\frac{18}{9}$, or 2.

This example was somewhat contrived, but it is surprising how many pupils quite happily cancel out expressions 'term by term', as the next example will show.

Example (30): Simplify $\frac{2x^2+7x-4}{2x^2+9x+4}$.

$$\frac{2x^2+7x-4}{2x^2+9x+4} = \frac{\overset{1}{\cancel{2x^2}}+7x-\overset{1}{\cancel{4}}}{\underset{1}{\cancel{2x^2}}+9x+\underset{1}{\cancel{4}}} = \frac{7x}{9x+2}$$

The pupil spotted the common terms of $2x^2$ on the top and bottom, so they were cancelled out to 1. The same method was applied when the pupil cancelled out the 4's from top and bottom.

We can only cancel out *factors* and not *individual terms* when simplifying expressions.

This is totally wrong - we **cannot** cancel out terms !

The correct working is :

$$\frac{2x^2+7x-4}{2x^2+9x+4} = \frac{(2x-1)(x+4)}{(2x+1)(x+4)}$$

$$= \frac{(2x-1)\cancel{(x+4)}}{(2x+1)\cancel{(x+4)}} = \frac{2x-1}{2x+1}$$

We *factorise* the quadratics on both the top and bottom lines, find the common factor of $(x + 4)$, and finally cancel it out.

Example (31): Simplify $\dfrac{4x^2-18}{x-3}$.

$$\frac{4x^2-18}{x-3} = \frac{4x^2}{x} - \frac{18}{3}$$

$$= 4x - 6 = 2(x-3)$$

The pupil had incorrectly 'decoupled' the terms from top and bottom to form two separate fractions
The correct method is:

$$\frac{4x^2-18}{x-3} = \frac{2(x+3)\cancel{(x-3)}}{\cancel{x-3}} = 2(x+3)$$

Simplification of equations can also lead to incorrect conclusions and 'lost' solutions.

Example (32): Solve the equation $2x^2 = 7x$.

$$2x^2 = 7x \rightarrow 2x = 7$$
$$\rightarrow x = 7/2$$

We have divided both sides of the equation by the variable x , but in so doing, we have lost a solution.

The correct working is:

$$2x^2 = 7x \rightarrow 2x^2 - 7x = 0$$
$$\rightarrow x(2x-7) = 0$$
$$\rightarrow x = 0,\ x = 7/2$$

We had lost the solution $x = 0$ in the incorrect working.

Careless cancellation can lead to more than just missing solutions.

Example (33): Here we have managed to 'prove' that 2 = 1.

i) Take two numbers a and b such that $a = b$.
ii) Multiply both sides by a and then subtract b^2 from both sides.

iii) Factorise, cancel and simplify.

iv) 2 = 1.

$$a = b \rightarrow a^2 = ab \rightarrow a^2 - b^2 = ab - b^2$$

$$\rightarrow (a-b)(a+b) = b(a-b)$$
$$\rightarrow (a+b) = b$$
$$\rightarrow b+b = b$$
$$\rightarrow 2b = b$$
$$\rightarrow 2 = 1$$

Something has gone wrong here, but what ?

The next brevity is even sillier.

Example (34): Solve the equation $3x = 2x$.

Divide both sides by x, and we have this weird result:

$$3x = 2x$$
$$\therefore 3 = 2$$

The correct result below holds the key to both this example and the last one !

$$3x = 2x \rightarrow 3x - 2x = 0 \rightarrow x = 0$$

Recall the previous example:

Everything is tickety-boo so far...

Factorising ....

Dividing both sides by $a - b$, but at the start of the sum we assumed $a = b$.

Hence we are dividing by zero.

$$a = b \rightarrow a^2 = ab \rightarrow a^2 - b^2 = ab - b^2$$

$$\rightarrow (a-b)(a+b) = b(a-b)$$
$$\rightarrow (a+b) = b$$
$$\rightarrow b+b = b$$
$$\rightarrow 2b = b$$
$$\rightarrow 2 = 1$$

In this example, the only solution to the equation $3x = 2x$ is $x = 0$ when done correctly. When we factored out x in the faulty working, we divided by zero.

Division by zero is an **undefined** operation – you simply cannot do it !

Example (35) : Solve the equation $\frac{x+1}{x+3}+\frac{x-2}{2x-1}=1.$

$$\frac{x+1}{x+3}+\frac{x-2}{2x-1}=1 \rightarrow (x+1)(2x-1)+(x-2)(x+3)=1$$

$$\rightarrow 2x^2+x-1+x^2+x-6=1$$

$$\rightarrow 3x^2+2x-7=1 \rightarrow 3x^2+2x-8=0$$

$$\rightarrow (3x-4)(x+2)=0 \rightarrow x=4/3,\ x=-2$$

The incorrect working above is a result of the pupil attempting to cross-multiply the expression. There might be a follow-through credit for attempting the solve the resulting quadratic.

If an equation can be written in the form $\frac{a}{b}=\frac{c}{d}$ where a, b, c and d are expressions and b and d are not zero, we *can* cross-multiply and write $ad = bc$.

The equation needs to be rewritten into a suitable form for cross-multiplication to work – here is the correct working

$$\frac{x+1}{x+3}+\frac{x-2}{2x-1}=1 \rightarrow \frac{x+1}{x+3}=1-\frac{x-2}{2x-1}$$

$$\rightarrow \frac{x+1}{x+3}=\frac{2x-1}{2x-1}-\frac{x-2}{2x-1} \rightarrow \frac{x+1}{x+3}=\frac{x+1}{2x-1}$$

$$\rightarrow (x+1)(2x-1)=(x+3)(x+1)$$

$$\rightarrow 2x^2+x-1=x^2+4x+3$$

$$\rightarrow 2x^2+x-1-x^2-4x-3=0$$

$$\rightarrow x^2-3x-4=0 \rightarrow (x-4)(x+1)=0$$

$$\rightarrow x=4,\ x=-1.$$

Example (36): Find the solution(s) of the simultaneous equations

$y = x^2 – 5x + 9$
$y = 2x - 3$

$\rightarrow$ simultaneous solution when $x^2-5x+9=2x-3$

$\rightarrow x^2-5x+9-2x-3=0$

$\rightarrow x^2-7x+6=0 \rightarrow (x-1)(x-6)=0$

x-coords of intersection pts are 1, 6

$x=1,\ y=-1$ (sub into $2x-3$)

$x=6,\ y=9$

The pupil had made a careless sign error here when subtracting $2x – 3$ from both sides.

$x^2 – 5x + 9 = (2x – 3)$

is equivalent to

$x^2 - 5x + 9 - (2x - 3) = 0$ or
$x^2 - 5x + 9 - 2x + 3 = 0$.

A minus sign outside brackets means reversing everything inside the brackets.

Correct version:

→ simultaneous solution
when $x^2 - 5x + 9 = 2x - 3$

→ $x^2 - 5x + 9 - 2x + 3 = 0$

→ $x^2 - 7x + 12 = 0$ → $(x-3)(x-4) = 0$

x-coords of intersection pts are 3, 4

$x = 3$, $y = 3$ (sub into $2x-3$)
$x = 4$, $y = 5$

Quadratic Quandaries.

Example (37): Solve the equation $x^2 - 12x + 15 = 0$, giving your answers in the form $a \pm \sqrt{b}$, where a and b are integers.

$$x^2 - 12x + 15 = 0$$

$$\rightarrow x = \frac{12 \pm \sqrt{144-60}}{2} = \frac{12 \pm \sqrt{84}}{2}$$

$$\rightarrow x = 6 \pm \sqrt{42} \quad x = 6+\sqrt{42},\ 6-\sqrt{42}.$$

The pupil made an error in the surd arithmetic: $\frac{\sqrt{84}}{2} = \frac{\sqrt{84}}{\sqrt{4}} = \sqrt{\frac{84}{4}} = \sqrt{21}$, not $\sqrt{42}$.

(See Example (20), part iv) for a similar case.)

$$x^2 - 12x + 15 = 0$$

$$\rightarrow x = \frac{12 \pm \sqrt{144-60}}{2} = \frac{12 \pm \sqrt{84}}{2}$$

$$\rightarrow x = 6 \pm \sqrt{21} \quad x = 6+\sqrt{21},\ 6-\sqrt{21}$$

Example (38): Solve the equation $x^2 - 6x + 8 = -1$.

The pupil forgot to rearrange the equation into the correct form, i.e. $x^2 - 6x + 9 = 0$.

Always have zero on the RHS when solving a quadratic equation by factorising.

$$x^2 - 6x + 8 = -1$$

$$\rightarrow (x-2)(x-4) = -1$$

$$\rightarrow x-2 = -1 \text{ or } x-4 = -1$$

$$\therefore x = 1 \text{ or } x = 3$$

The correct working is :

$$x^2 - 6x + 8 = -1$$

$$\rightarrow x^2 - 6x + 9 = 0$$

$$\rightarrow (x-3)^2 = 0$$

$$\rightarrow x = 3.$$

Infuriating Inequalities.

Example (39) : Solve the inequality $10 - x > 3 + 2x$.

$10 - x > 3 + 2x$	$10 - x > 3 + 2x$
$-x > 2x - 7$	$-x > 2x - 7$
$-3x > -7$	$-3x > -7$
✗ $x > -7/3$	✓ $x < -7/3$

The pupil forgot to reverse the inequality sign when dividing by (-3) in the last step !
The correct answer is shown on the right.

Example (40) :

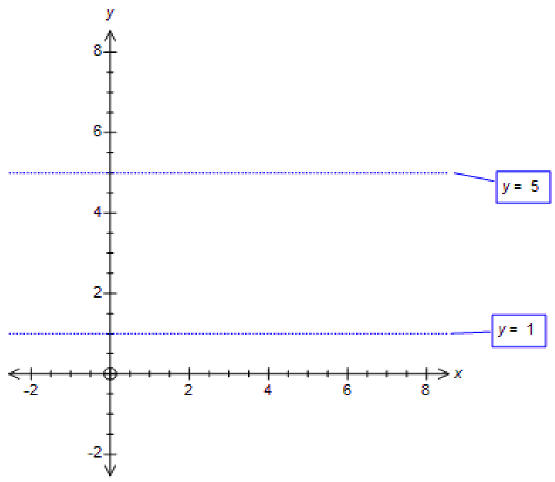
i) Find the region of the plane satisfying the three inequalities

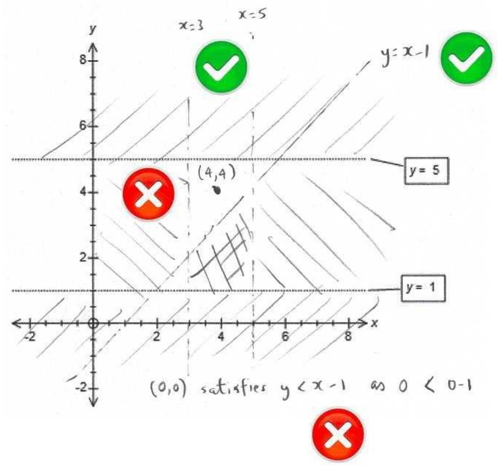
$1 < y < 5$; $3 < x < 5$; $y < x - 1$

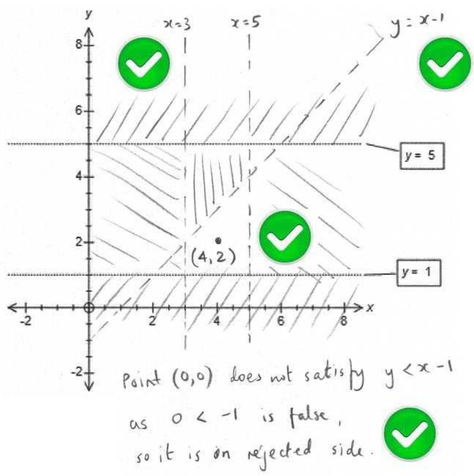
shading out the rejected regions.

(The lines $y = 1$ and $y = 5$ have been plotted.)

ii) In that required region, plot any point(s) (x , y) where x and y are integers.





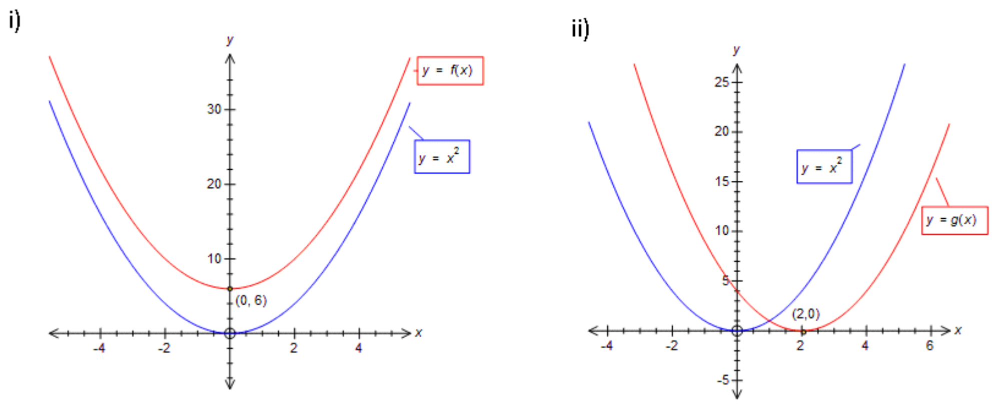


In the case of the faulty answer on the left, the pupil had rejected the region on the wrong side of the inequality $y < x - 1$, by mistakenly assuming $0 < 0\text{-}1$. The corrected answer is on the right.

(Both examples are correct, though, in identifying the inequalities as strict, hence no points on the boundary lines had been included.)

Treacherous Transformations.

Example (41): Two transformations of the graph $y = x^2$ are shown below.
Find the equations of $y = f(x)$ and $y = g(x)$.



(i) origin translated to (0,6)
$\rightarrow$ translation $\begin{pmatrix}0\\6\end{pmatrix}$, so $f(x) = x^2 + 6$

(ii) Origin translated to (2,0)
$\rightarrow$ translation $\begin{pmatrix}2\\0\end{pmatrix}$, so $g(x) = (x+2)^2$

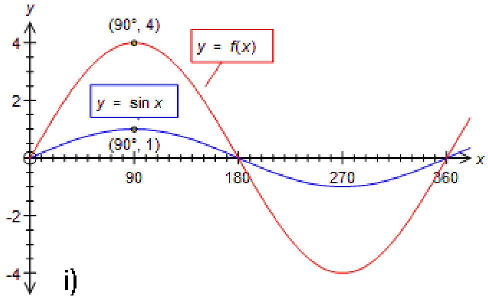
The x-translation appears to work the wrong way at first, but substituting $x = 2$ in $y = (x+2)^2$ gives an answer of $y = 16$, not 0.

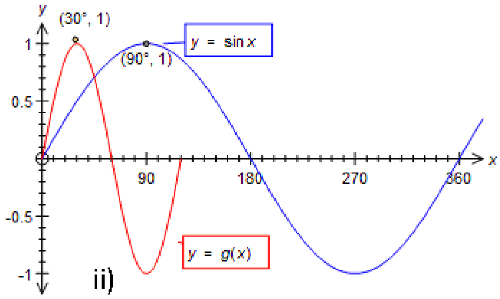
Substituting $x = -2$ in $y = (x+2)^2$ gives an answer of $y = 0$ as per the graph.
Correct is:

(i) origin translated to (0,6)
$\rightarrow$ translation $\begin{pmatrix}0\\6\end{pmatrix}$, so $f(x) = x^2 + 6$

(ii) Origin translated to (2,0)
$\rightarrow$ translation $\begin{pmatrix}2\\0\end{pmatrix}$, so $g(x) = (x-2)^2$

Example (42): Two transformations of the graph $y = \sin x$ are shown below
Find the equations of $y = f(x)$ and $y = g(x)$.





(i) Point $(90°, 1)$ transformed to $(90°, 4)$
$\rightarrow$ y-stretch of scale factor 4, so $f(x) = 4 \sin x$

(ii) Point $(90°, 1)$ transformed to $(30°, 1)$
$\rightarrow$ x-stretch of scale factor $\frac{1}{3}$, so $g(x) = \sin\left(\frac{x}{3}\right)$

Just like the x-translation, the x-stretch also appears to work the wrong way at first. This x-stretch has a scale factor of $\frac{1}{3}$, so its equation is $y = \sin(3x)$, not $y = \sin(\frac{1}{3}x)$.
Let $x = 30°$, then $\sin(3x) = \sin 90° = 1$ as can be verified by the graph.

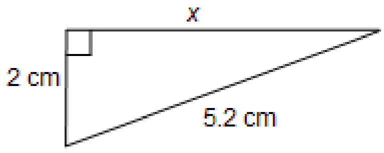
Correct is:

(i) Point $(90°, 1)$ transformed to $(90°, 4)$
$\rightarrow$ y-stretch of scale factor 4, so $f(x) = 4 \sin x$

(ii) Point $(90°, 1)$ transformed to $(30°, 1)$
$\rightarrow$ x-stretch of scale factor $\frac{1}{3}$, so $g(x) = \sin(3x)$

Trigonometric Trauma.

Example (43): Find the length of side *x* in the right-angled triangle shown.

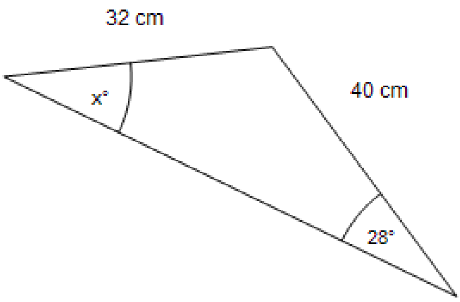


$$x^2 = 5.2^2 + 2^2 = 31.04 \rightarrow x = \sqrt{31.04} \rightarrow x = 5.57 \text{ cm}$$

The pupil failed to realise that side *x* was not the hypotenuse. The square of the 2 cm side should have been *subtracted* from that of the hypotenuse, i.e. the 5.2 cm side.

$$x^2 = 5.2^2 - 2^2 = 23.04 \rightarrow x = \sqrt{23.04} = 4.8 \text{ cm}$$

Example (44): Find the angle labelled *x* in the triangle shown.



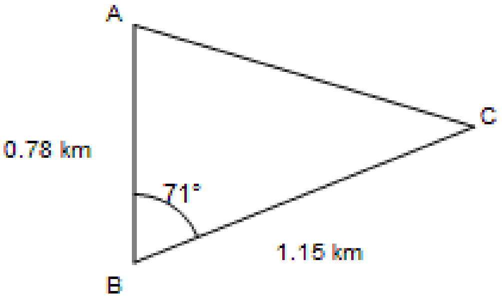
$$\frac{x}{\sin 40^\circ} = \frac{32}{\sin 28^\circ} \rightarrow x = \frac{32 \sin 40^\circ}{\sin 28^\circ} = 43.8^\circ$$

The pupil had mixed up the sides and angles here. We are given a side of 40 cm, not an angle of 40°.
Correct working :

$$\frac{\sin x}{40} = \frac{\sin 28^\circ}{32} \rightarrow \sin x = \frac{40 \sin 28^\circ}{32} \rightarrow \sin x = 0.587$$

$$\rightarrow x = 35.9^\circ.$$

Example (45): *A*, *B* and *C* are the corners of a triangular field.
Find i) the area of the field;   ii) the perimeter of the field.



$$\text{Area} = \tfrac{1}{2}\,AB.BC \sin 71^\circ = \tfrac{1}{2} \times 0.78 \times 1.15 \times \sin 71^\circ \quad 0.369 \text{ km}^2.$$

To find AC, use cosine rule. $(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos 71^\circ$

$$\rightarrow AC^2 = 0.78^2 + 1.15^2 - (2 \times 0.78 \times 1.15 \cos 71^\circ) = 1.347$$

$$\therefore \text{Perimeter} = 0.78 + 1.15 + 1.347 = 3.28 \text{ km}$$

One step had been carelessly omitted from the working. The pupil had forgotten to take the square root when calculating the length *AC.*

Correct version:

$$\text{Area} = \tfrac{1}{2}\,AB.BC \sin 71^\circ \quad \tfrac{1}{2} \times 0.78 \times 1.15 \times \sin 71^\circ = 0.369 \text{ km}^2.$$

To find AC, use cosine rule. $(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos 71^\circ$

$$\rightarrow AC^2 = 0.78^2 + 1.15^2 - (2 \times 0.78 \times 1.15 \cos 71^\circ) = 1.347$$

$$\rightarrow AC = \sqrt{1347} = 1.1605$$

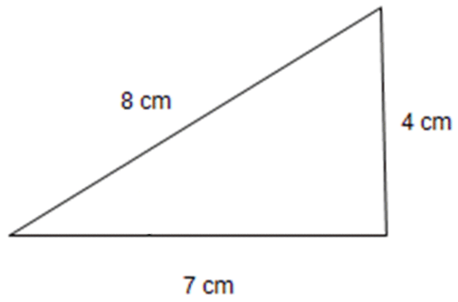
$$\therefore \text{Perimeter} = 0.78 + 1.15 + 1.16 = 3.09 \text{ km.}$$

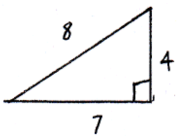
When a question is illustrated by a diagram, then the written annotations should be assumed to be correct, but that the diagram itself is *not* drawn accurately.

Example (46):

i) Show that the area of the triangle shown is 14.0 cm^2 to 3 significant figures.

You must show all your working.

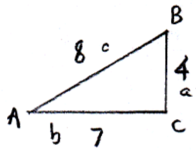




Area $= \frac{1}{2}$ base $\times$ height

$= \frac{1}{2} \times 7 \times 4 = 14.0$ cm^2

This 'solution' has one lethal flaw – the pupil assumed that, because the triangle *looked* right-angled, it actually *was* right-angled. (The sides do not satisfy Pythagoras' Theorem, as $4^2 + 7^2 = 65$, but $8^2 = 64$.)



$\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{16+49-64}{2 \times 4 \times 7} = \dfrac{1}{56}$

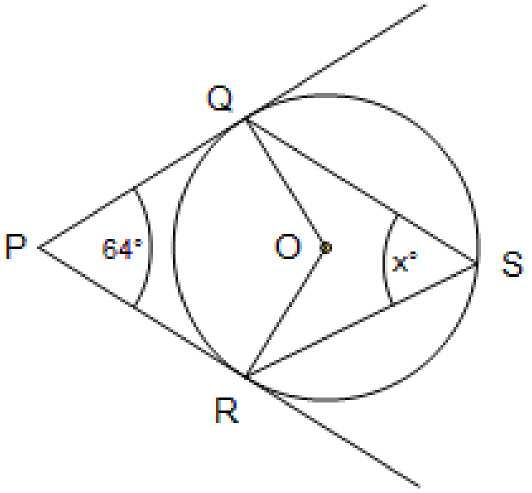
$C = 88.98°$ Area $= \frac{1}{2} \times 4 \times 7 \times \sin 88.98°$

$= 13.998$ cm^2 $= 14.0$ cm^2 to 3 s.f.

Because we are not dealing with a right-angled triangle, we need to use the cosine formula and the general triangle area formula to answer the question.

Vicious Circles.

Example (47): Two tangents originating from point P touch a circle with centre O at points Q and R. Calculate angle QSR, labelled *x* in the diagram.
Give reasons for each stage of your working.



Calculate angle *x* , giving reasons for each step in your working.

$\angle ROQ = 180° - 64° = 116°$ (opposite angles of cyclic quadrilateral add to 180°)

$\angle QSR = x = \frac{116°}{2} = 58°$ (angle at centre is double the angle at circumference)

The pupil obtained the correct value for the angle *x*, but did not explain clearly why the quadrilateral PQOR was cyclic.

Correct reasoning below:

$\angle$s PQO, PRO = 90° (tangents and radii meet at 90°)

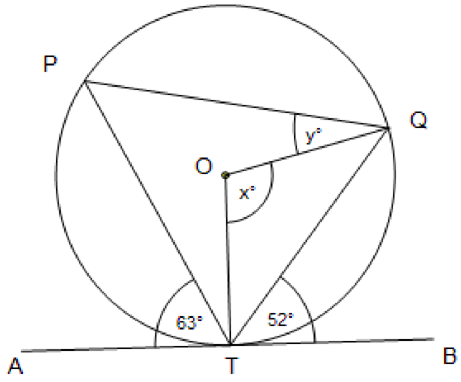
$\therefore \angle ROQ = 360 - (180 + 64) = 116°$ (angle-sum of quadrilateral = 360°)

$\angle QSR = x = \frac{116°}{2} = 58°$ (angle at centre is double the angle at circumference)

Interestingly, any quadrilateral bounded by two tangents and two radii is cyclic because it has an opposite pair of right angles !

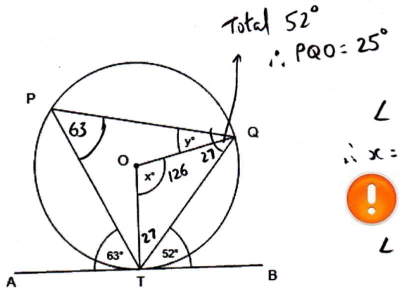
Example (48) : Line AB is a tangent at point T to the circle centred on O. Points P and Q lie on the circle's circumference.

Calculate angles *x* and *y*, showing each step in your working.



The pupil had confused the alternate segment theorem with that of alternate angles in parallel lines, despite the fact that lines AB and PQ are not parallel.

Other errors had crept in as a result, such as angle OTB working out at 27° + 52° or 79°, instead of 90°.



$\angle$ TPQ = 63° (alternate angles)

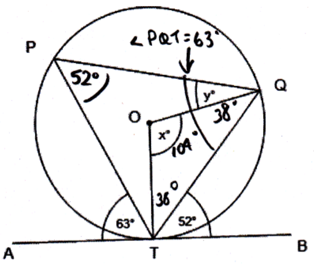
∴ x = $\angle$TOQ = 126° (angle at centre is double the angle at circumference)

$\angle$ TQO = 27° ($=\frac{1}{2}(180-126)^\circ$), as Δ TOQ is isosceles.

$\angle$ PQT = 52° (alternate angles)

∴ $\angle$PQO = y = 52 − 27 = 25°

The correct working is shown below.



$\angle$ TPQ = 52° (alternate segment)

∴ x = $\angle$TOQ = 104°

(angle at centre = double the angle at circumference

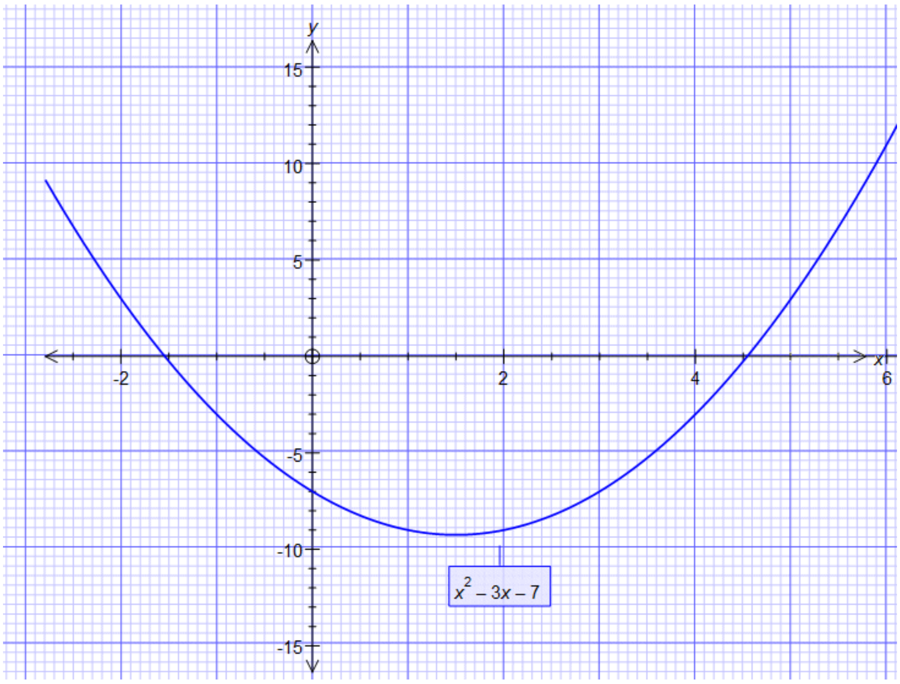
$\angle$ OTQ, OQT = 38° (Δ TOQ is isosceles)

$\angle$ PQT = 63° (alternate segment)

∴ y° + 38° = 63° → y = 25°

(Examples 49-60: more algebra, shape and space pending)

Example (49) : The graph of $y = x^2 - 3x - 7$ is shown below.



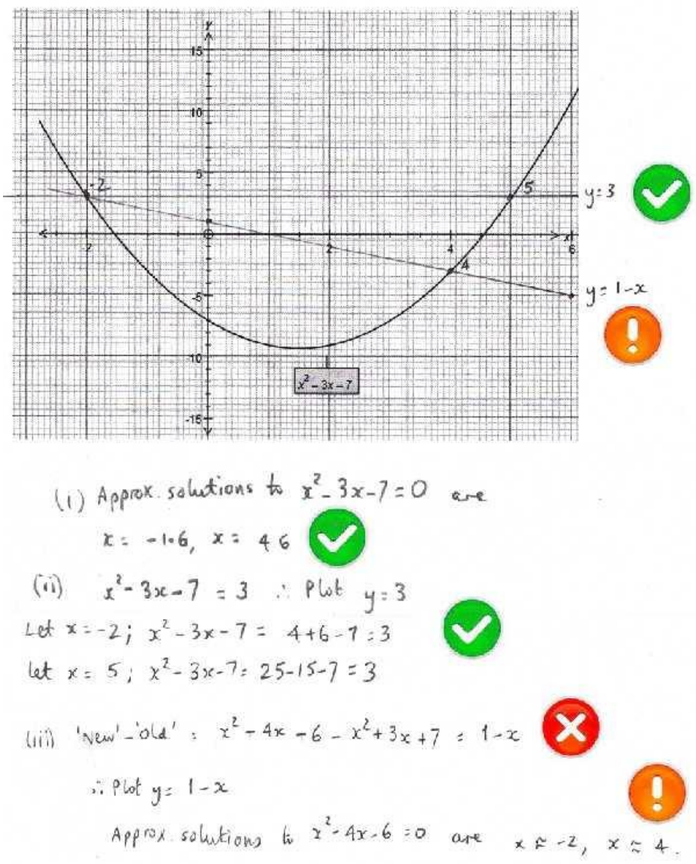
i) Using the graph, find the approximate solutions to $x^2 - 3x - 7 = 0$.

ii) By plotting a suitable straight line, find the solutions to $x^2 - 3x - 7 = 3$.
Verify your results *algebraically*.

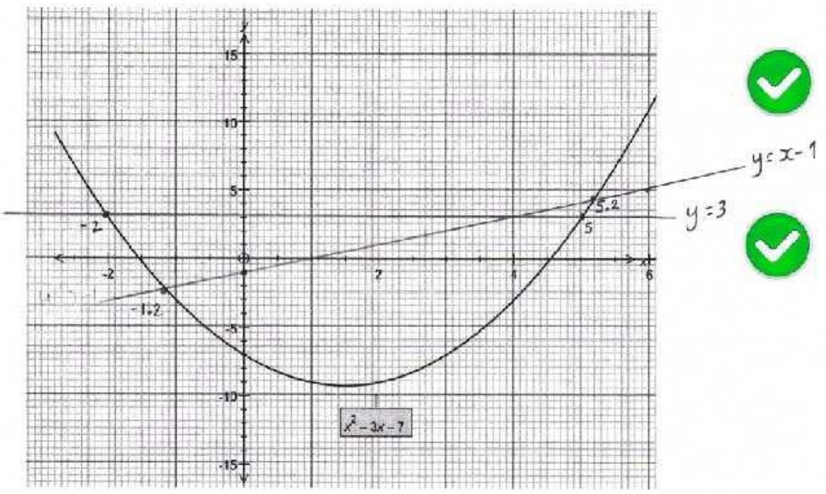
iii) By plotting another suitable straight line, find the approximate solutions to $x^2 - 4x - 6 = 0$.

Parts i) and ii) were answered correctly, but in part iii) the pupil had subtracted the quadratics in the wrong order, and ended up with an incorrect equation for the straight line, namely $y = 1 - x$ instead of $y = x - 1$.

The given line equation is in fact the correct one multiplied by -1.



The correct working is as follows :



(i) Approx. solutions to $x^2 - 3x - 7 = 0$ are

$x = -1.6$, $x = 4.6$

(ii) $x^2 - 3x - 7 = 3 \rightarrow x^2 - 3x - 10 = 0$ ∴ Plot $y = 3$

$\rightarrow (x+2)(x-5) = 0$ ∴ $x = -2$, $x = 5$.

(iii) 'old' : $x^2 - 3x - 7$, 'New' : $x^2 - 4x - 6$

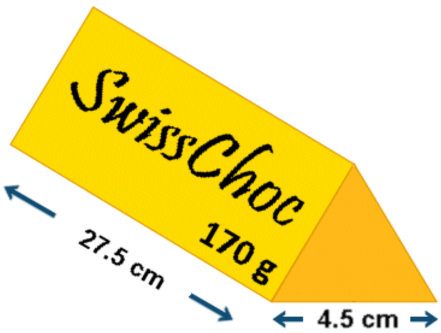
'old' – 'New' : $x^2 - 3x - 7 - x^2 + 4x + 6 = x - 1$

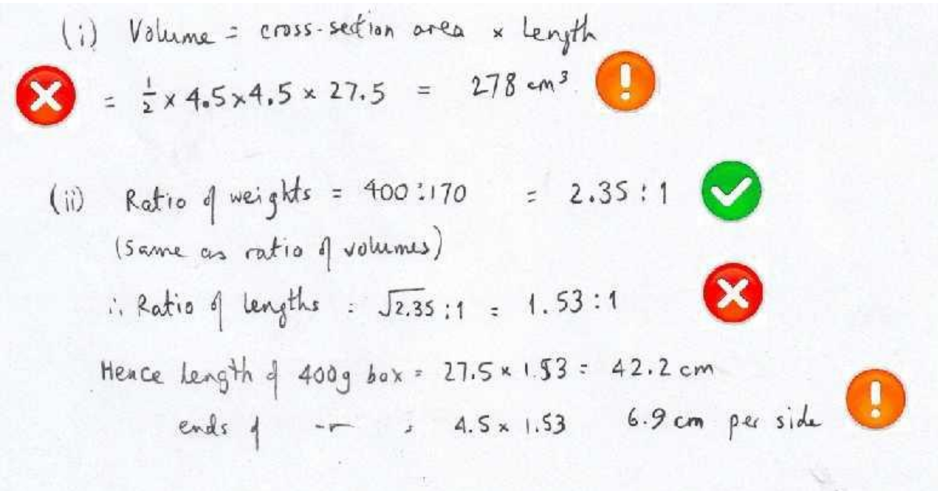
∴ Plot $y = x - 1$

Approx. solutions to $x^2 - 4x - 6 = 0$ are $x \approx -1.2$, $x \approx 5.2$

Example (50) : i) Calculate the volume of the container for this SwissChoc bar. It is in the shape of a prism, and the end faces are equilateral triangles.

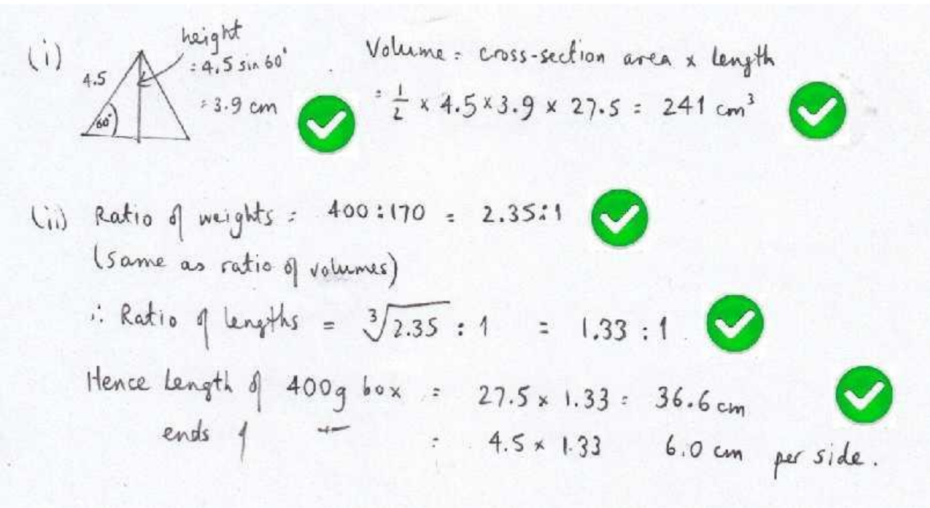
ii) SwissChoc bars are also sold in 400g packs, mathematically similar in all respects to the 170g pack shown. Calculate the corresponding lengths of the 400g pack.



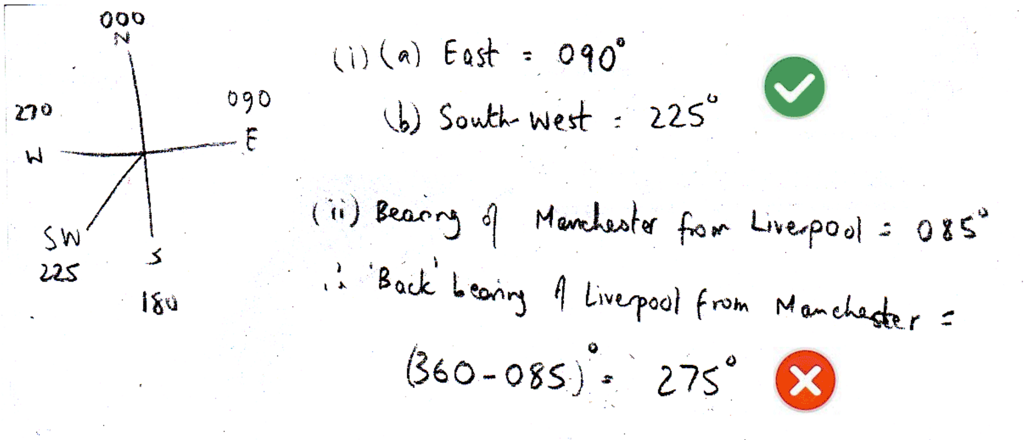


The pupil had made two errors here. Firstly, the sloping height of the triangular end face was used instead of the perpendicular height, and secondly, the ratio of the lengths of the two packs was calculated as the *square root* instead of the *cube root* of the weight ratio.

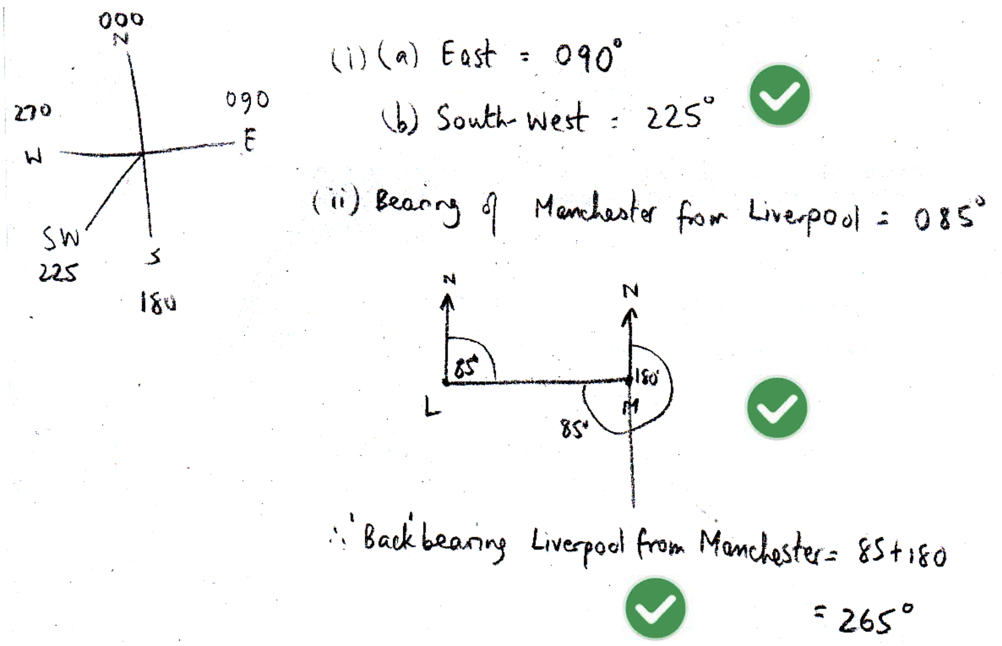
The correct working is shown below.



Example (51):
i) Express the following compass points in bearing notation : a) East b) South-West
ii) The bearing of Manchester Airport from Liverpool Airport is 085° . Calculate the bearing of Liverpool Airport from Manchester Airport.



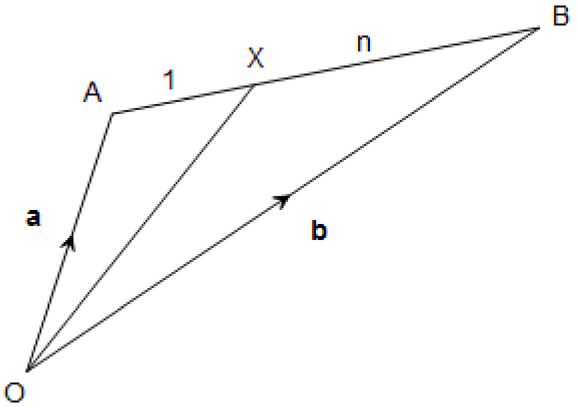
The pupil had subtracted the bearing of Manchester from Liverpool from 360°, instead of adding 180°. A diagram (as below) would have prevented the error.



Examples (52) to (58) for shape and space currently pending.

Vector Vagaries.

Example (59): In the diagram, $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b.**



Point X lies on line AB such that the ratio AX:XB = 1 : n.

i) Express vectors $\overrightarrow{AB}$, $\overrightarrow{AX}$ and $\overrightarrow{OX}$ in terms of **a** and **b**.

ii) We are then told that point X is one-third of the way from A to B.

Find the particular vector expression $\overrightarrow{OX}$ in terms of **a** and **b**.

i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \underline{a} + \underline{b}$

$\overrightarrow{AX} = \frac{1}{1+n}\overrightarrow{AB} = \left(\frac{1}{1+n}\right)\underline{a} + \left(\frac{1}{1+n}\right)\underline{b}$

$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \underline{a} + \left(\frac{1}{1+n}\right)\underline{a} + \left(\frac{1}{1+n}\right)\underline{b}$

$= \left(\frac{2+n}{1+n}\right)\underline{a} + \left(\frac{1}{1+n}\right)\underline{b}$

(ii) As $AX : XB = 1:2$, $n = 2$.

$\overrightarrow{OX} = \frac{4}{3}\underline{a} + \frac{1}{3}\underline{b}$

The pupil began badly by failing to realise that vector **a** should have been subtracted and not added. The correct working is shown below.

(i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\underline{a} + \underline{b} = \underline{b} - \underline{a}$

$\overrightarrow{AX} = \frac{1}{1+n}\overrightarrow{AB} = \left(\frac{1}{1+n}\right)\underline{b} - \left(\frac{1}{1+n}\right)\underline{a}$

$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \underline{a} + \left(\frac{1}{1+n}\right)\underline{b} - \left(\frac{1}{1+n}\right)\underline{a}$

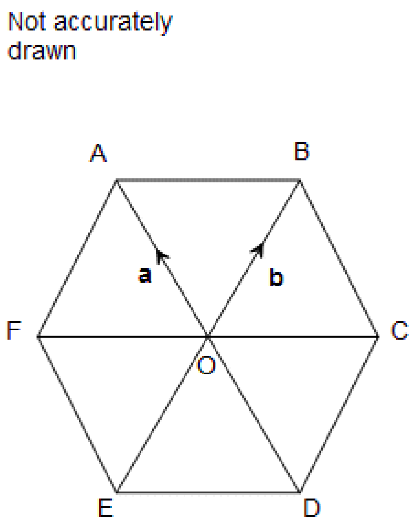
$= \left(\frac{n}{1+n}\right)\underline{a} + \left(\frac{1}{1+n}\right)\underline{b}.$

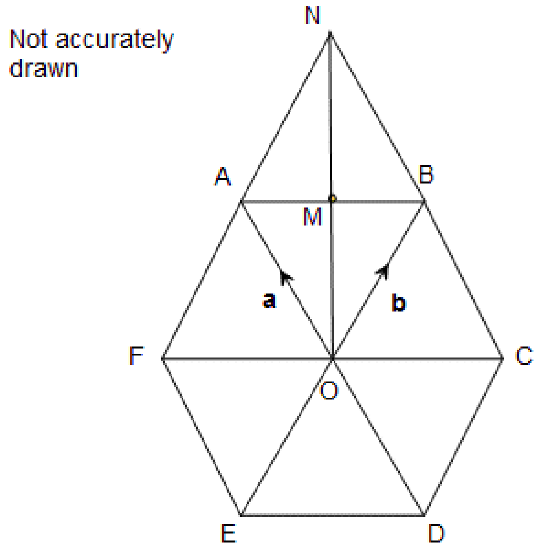
(ii) As $AX : XB = 1:2$, $n = 2$

$\overrightarrow{OX} = \frac{2}{3}\underline{a} + \frac{1}{3}\underline{b}$

Example (60): *ABCDEF* is a regular hexagon with centre *O*.

$\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b.**





i) Name one vector in the diagram equal to **a**, and another vector equal to –**b**.

ii) Find vector expressions (in terms of **a** and **b**) for $\overrightarrow{OC}$, $\overrightarrow{OF}$ and $\overrightarrow{AB}$

iii) *M* is the midpoint of *AB*. Express $\overrightarrow{OM}$ in terms of **a** and **b**.

iv) *FA* and *CB* are extended to twice their length to meet at point *N*. Express $\overrightarrow{ON}$ in terms of **a** and **b**.
v) From the last result, how are points *O*, *M* and *N* related ?

The pupil had added vector **a** instead of subtracting it in part (ii), and also failed to spot the scalar multiple relationship between the results in iii) and iv) despite obtaining the correct result.

The corrected answer is on the next page.

(i) $\overrightarrow{EF} = \underline{a}$, $\overrightarrow{OE} = -\underline{b}$

(ii) $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$ $\overrightarrow{BC}$ parallel & equal to OA,

x $\therefore \overrightarrow{OC} = \underline{b} + \underline{a}$

y $\overrightarrow{OF} = -\overrightarrow{OC}$ (different direction) $= -\underline{b} - \underline{a}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\underline{a} + \underline{b}$

(iii) $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \underline{a} - \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$

$= \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$.

(iv) $\overrightarrow{FA} = \underline{b}$, so $\overrightarrow{FN} = 2\underline{b}$ and $\overrightarrow{AN} = \underline{b}$

$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \underline{a} + \underline{b}$

(v) ?

(i) $\overrightarrow{EF} = \underline{a}$, $\overrightarrow{OE} = -\underline{b}$

(ii) $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$ $\quad \overrightarrow{BC}$ parallel to OA, opposite direction

$\therefore \overrightarrow{OC} = \underline{b} - \underline{a}$; $\overrightarrow{OF} = -\overrightarrow{OC} = \underline{a} - \underline{b}$

$\overrightarrow{AB} = \overrightarrow{OC} = b - \underline{a}$

(iii) $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \underline{a} - \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$

$= \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$.

(iv) $\overrightarrow{FA} = \underline{b}$, so $\overrightarrow{FN} = 2\underline{b}$ and $\overrightarrow{AN} = \underline{b}$

$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \underline{a} + \underline{b}$

(v) $\overrightarrow{OM} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$; $\overrightarrow{ON} = \underline{a} + \underline{b}$

$\overrightarrow{OM}$ is scalar multiple of $\overrightarrow{ON}$ (half of it)

$\therefore$ Points O, M, N are collinear with M the midpoint of ON.

Example (61): Here is some information about how many texts Year 11 pupils sent over a certain week:

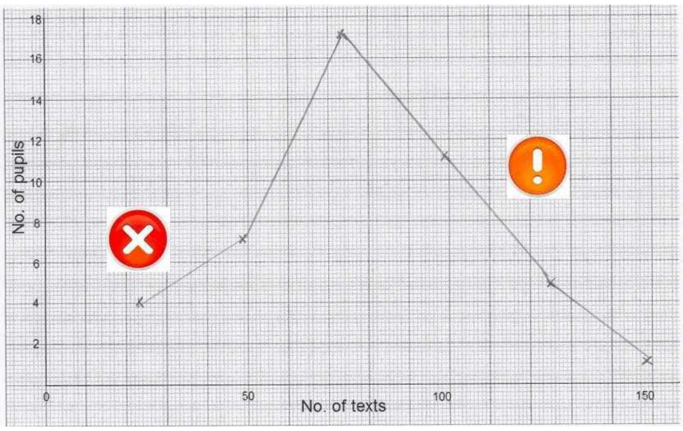
No. of texts	0-24	25-49	50-74	75-99	100-124	125-149
No. of pupils	4	7	17	11	5	1

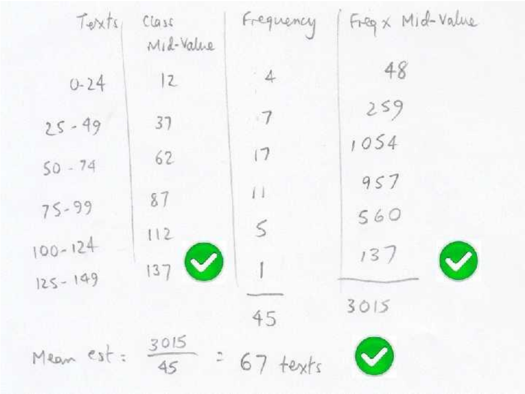
i) Express this data on a frequency polygon.

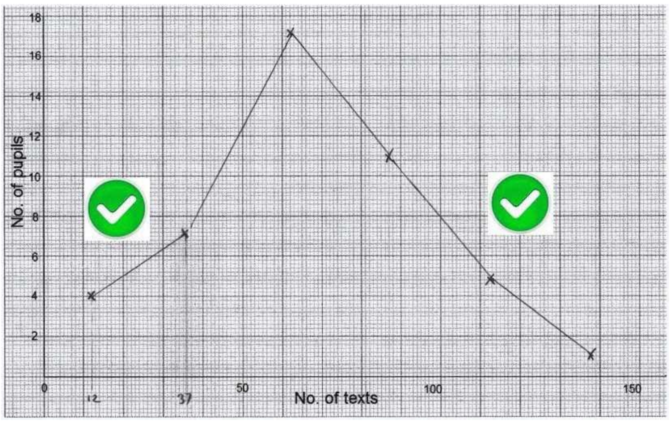
ii) Estimate the mean number of texts sent by the pupils.

No. of texts	0-24	25-49	50-74	75-99	100-124	125-149
No. of pupils	4	7	17	11	5	1

The pupil had drawn the frequency polygon to the correct basic shape, but the cross marks had been aligned with the *ends* of the class intervals (24, 49 ...) instead of their *midpoints* (12, 37 ...) .
.
The calculations of the estimated mean, however, were quite correct.



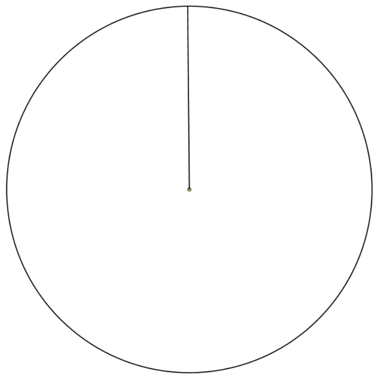




The corrected frequency polygon is shown on the right

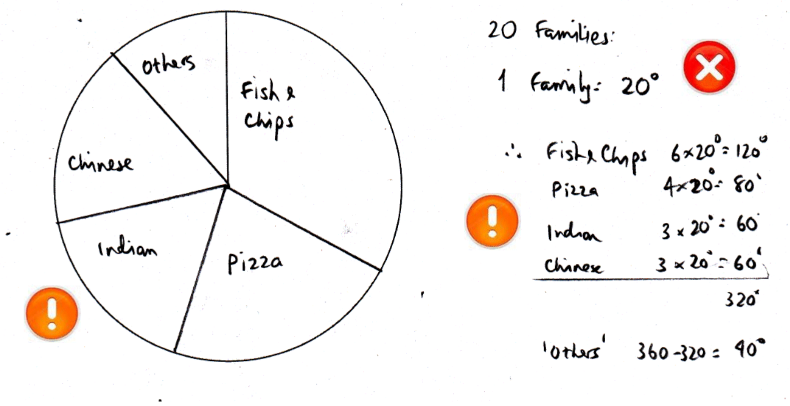
Example (62): Twenty families in a street were asked about their favourite type of takeaway food.
6 said fish and chips, 4 said pizza, 3 Indian, 3 Chinese and the rest 'others'.

Draw a pie chart to represent this information.

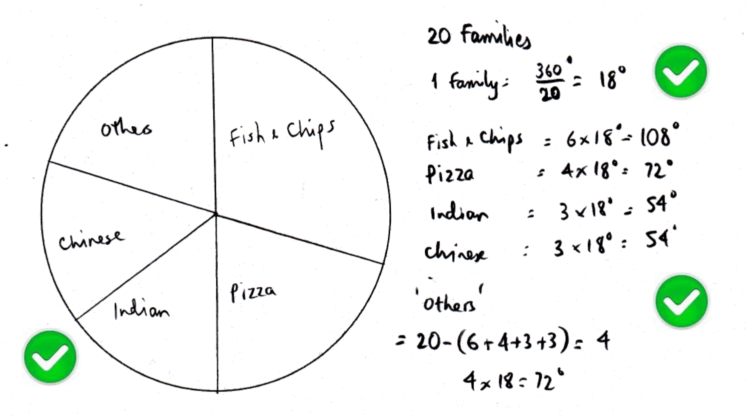


The pupil had miscalculated the angle corresponding to each 'slice' of the pie.

If 20 families are represented by 360°, one family is represented by 18°, not 20°.



The correct pie chart is shown below.



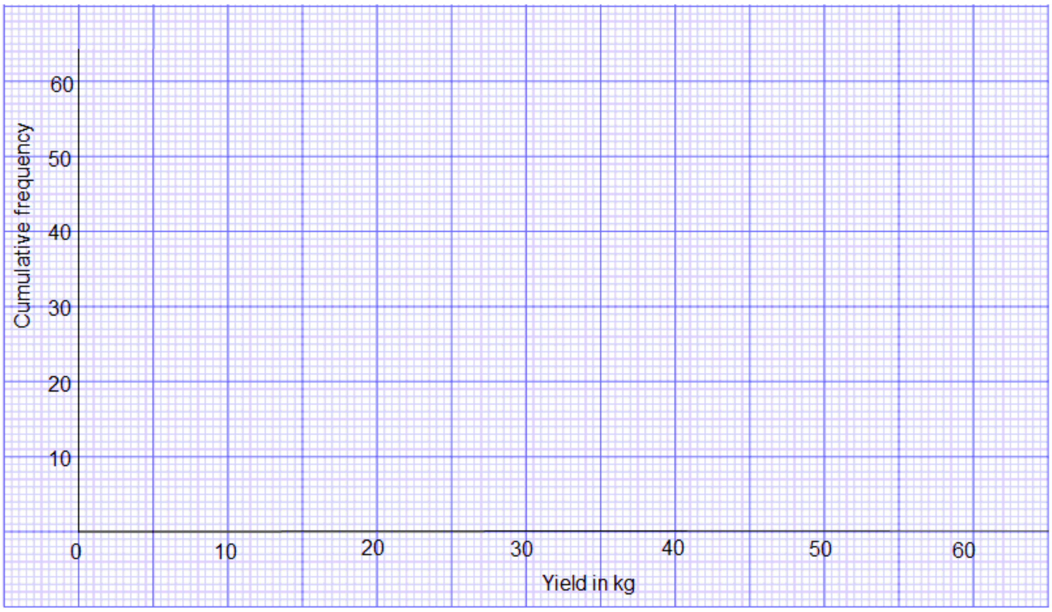
Example (63): A fruit grower recorded the yields of apples for the 64 trees in his orchard.

Yield (kg), w	$10 \le w < 20$	$20 \le w < 25$	$25 \le w < 30$	$30 \le w < 35$	$35 \le w < 40$	$40 \le w < 60$
No. of trees	4	9	21	18	8	4

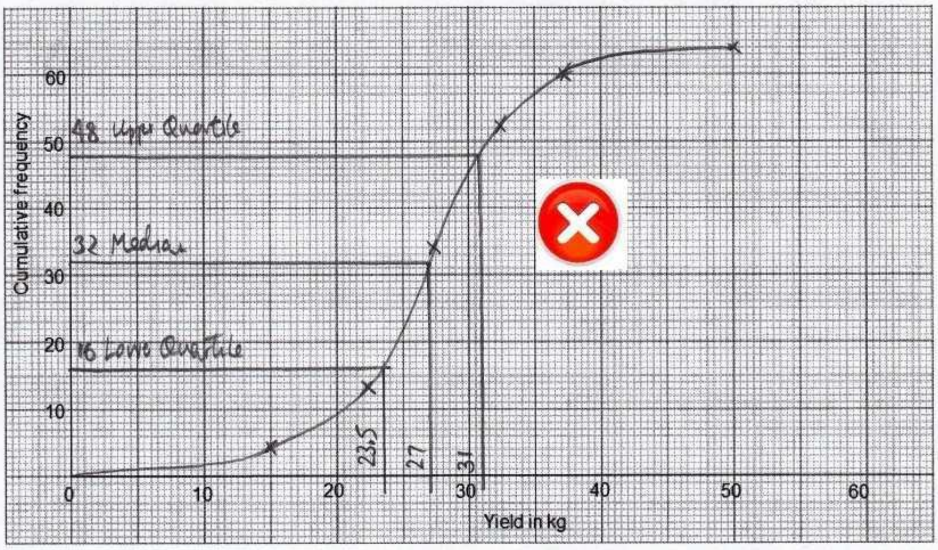
i) Complete the cumulative frequency table.

Yield (kg), w	$10 \le w < 20$	$10 \le w < 25$	$10 \le w < 30$	$10 \le w < 35$	$10 \le w < 40$	$10 \le w < 60$
No. of trees	4	13				

ii) Plot the cumulative frequency graph and estimate a) the median and b) the inter-quartile range.



Yield (kg), w	$10 \le w < 20$	$10 \le w < 25$	$10 \le w < 30$	$10 \le w < 35$	$10 \le w < 40$	$10 \le w < 60$
No. of trees	4	13	34	52	60	64



Median $\approx$ 23.5 kg

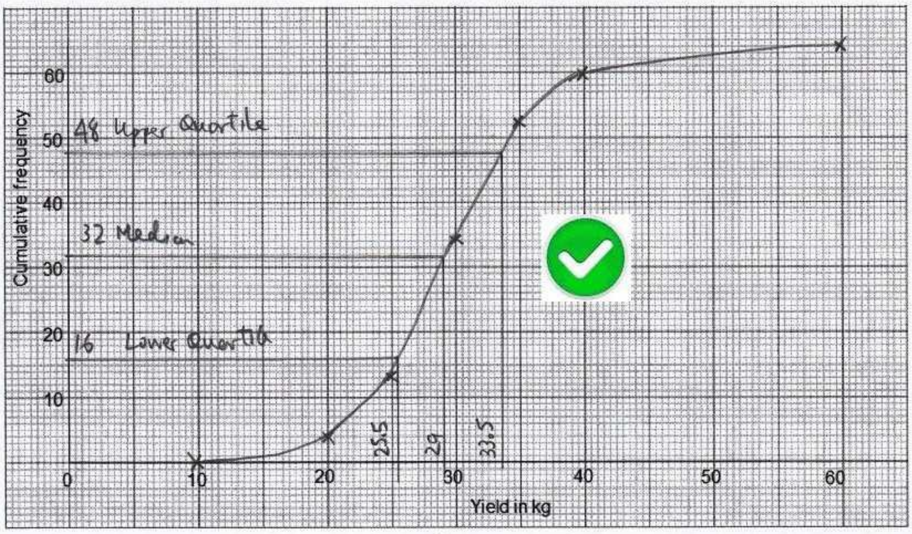
Interquartile range: 31 - 23.5 = 7.5kg

The pupil had plotted the cumulative frequencies against the *midpoints* of the class intervals instead of their *endpoints.*

As a result, the median and quartiles had been estimated at rather less than their true values.

The correct working is shown on the next page.

Yield (kg), w	$10 \le w < 20$	$10 \le w < 25$	$10 \le w < 30$	$10 \le w < 35$	$10 \le w < 40$	$10 \le w < 60$
No. of trees	4	13	34	52	60	64

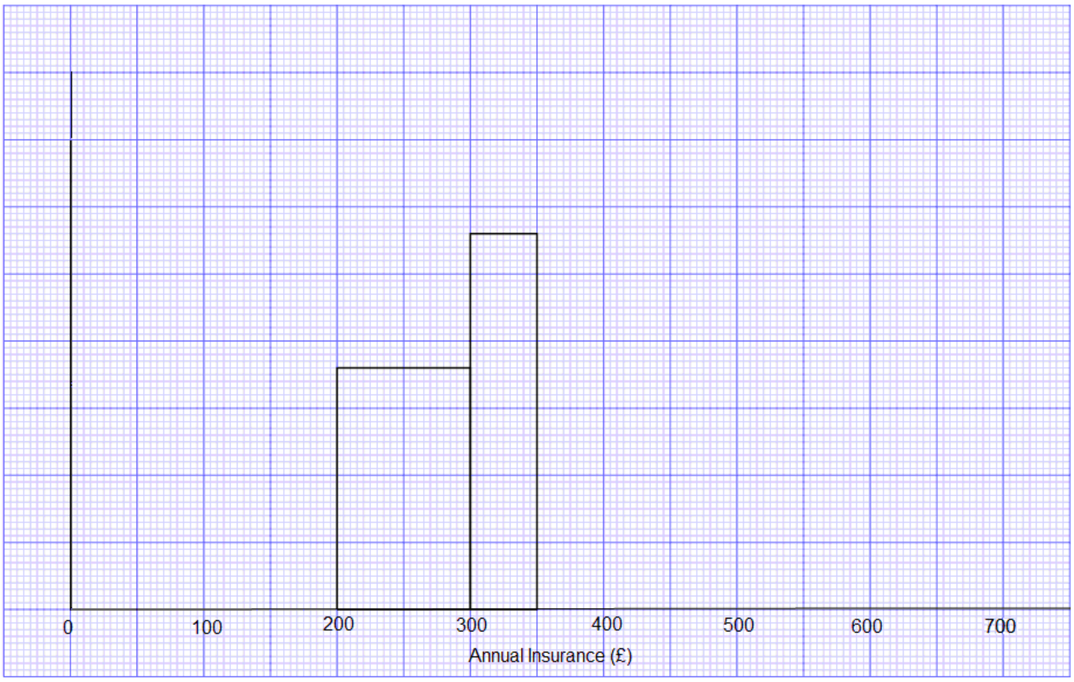


Median $\approx$ 29kg

Inter-quatile range $\approx 33.5 - 25.5 = 8$ kg

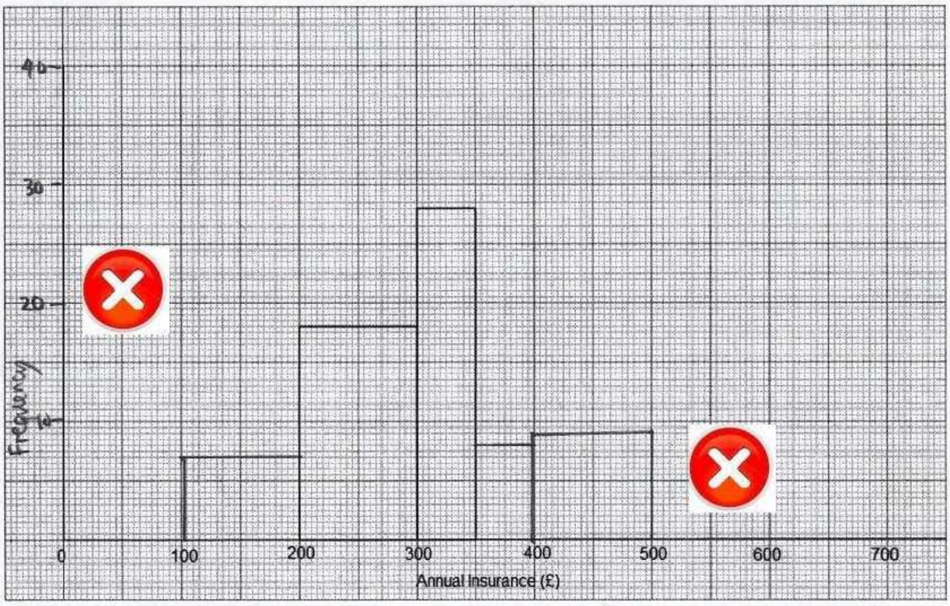
Example (64): A sample of 60 people on an estate was asked how much they paid each year on their home insurance.

Annual Insurance (£), x	Freq.
$100 \le x < 200$	7
$200 \le x < 300$	18
$300 \le x < 350$	
$350 \le x < 400$	8
$400 \le x < 500$	9
$500 \le x < 700$	



i) Use the information in the table to complete the histogram.
ii) Use the information in the histogram to complete the table.

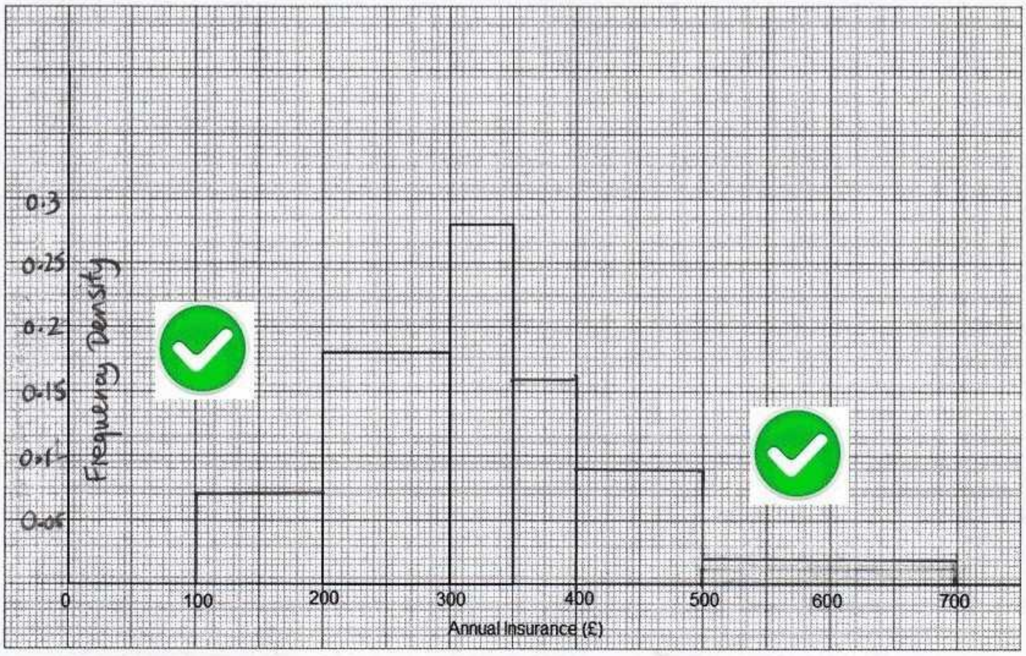
Annual Insurance (£), x	Freq.
$100 \le x < 200$	7
$200 \le x < 300$	18
$300 \le x < 350$	28
$350 \le x < 400$	8
$400 \le x < 500$	9
$500 \le x < 700$	?



The pupil had treated the histogram as if it were a bar chart, without any understanding of frequency density and area.

As a result, the sum of the frequencies came out as 67 (even without the missing class), and the shape of the 'histogram' came out totally wrong.

Annual Insurance (£), x	Freq.	Class width	Freq. Density
$100 \le x < 200$	7	100	0.07
$200 \le x < 300$	18	100	0.18
$300 \le x < 350$	14	50	0.28
$350 \le x < 400$	8	50	0.16
$400 \le x < 500$	9	100	0.09
$500 \le x < 700$	4	200	0.02



The true histogram is shown here, with frequency densities correctly calculated and the frequencies represented by the correct areas.

Example (65): i) A card is drawn at random from a standard 52-pack of 4 suits and 13 ranks.

What is the probability of it being a) a Jack; b) the Ace of Hearts; c) an Ace or a Two; d) any rank other than a Five; e) a Club or a Queen ?

(i) (a) $P(\text{Jack}) = \frac{1}{13}$, (b) $P(\text{Ace of Hearts}) = \frac{1}{52}$, (c) $P(\text{A or 2}) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$

(d) $P(\text{Five}) = \frac{1}{13}$ $\therefore$ $P(\text{not Five}) = 1 - \frac{1}{13} = \frac{12}{13}$

(e) $P(\text{Club or Queen}) = P(\text{Club}) + P(\text{Queen}) = \frac{1}{4} + \frac{1}{13} = \frac{17}{52}$

The pupil slipped up in the last part, because drawing a Club and drawing a Queen are *not* mutually exclusive. You could draw the Queen of Clubs !

(e) $P(\text{Club}) = \frac{13}{52}$; $P(\text{Queen}) = \frac{1}{13}$, but deduct $\frac{1}{52}$ for Queen of Clubs

$\therefore$ $P(\text{Club or Queen}) = \frac{13}{52} + \frac{1}{13} - \frac{1}{52} = \frac{4}{13}$. (Not mutually exclusive!)

There are 13 Clubs in the pack and 4 Queens, but that would mean counting the Queen of Clubs twice, and hence we must compensate by subtraction.

.
ii) A game involves a toss of a coin and the drawing of a card. What is the probability of throwing a head and drawing a King ?

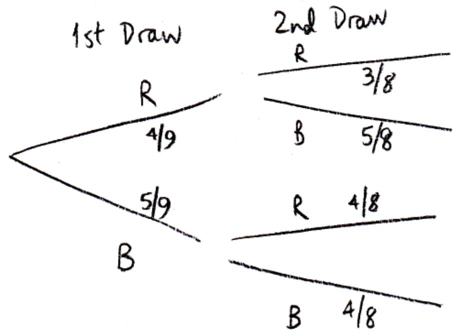
(ii) $P(\text{Head}) = \frac{1}{2}$ $P(\text{King}) = \frac{1}{13}$ $\therefore$ $P(\text{Head, King}) = \frac{1}{2} + \frac{1}{13} = \frac{15}{26}$

The toss of the coin and the draw of the card are *independent* events, so we *multiply* the probabilities and not *add* them !

(ii) $P(\text{Head}) = \frac{1}{2}$ $P(\text{King}) = \frac{1}{13}$ $\therefore$ $P(\text{Head, King}) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

Example (66): i) A box contains 4 red cards and 5 blue cards. A game involves drawing two cards out of the box one at a time without replacing the cards back in the box.

What is the probability of i) drawing two red cards; ii) drawing at least one blue card ?



$R,R = \frac{\cancel{4}^{1}}{\cancel{9}_{3}} \times \frac{\cancel{3}^{1}}{\cancel{8}_{2}} = \frac{1}{6}$

$R,B = 4/9 \times 5/8 = 20/72 = 5/18$

$B,R = 5/9 \times 4/8 = 20/72 = 5/18$

$B,B = 5/9 \times 4/8 = 20/72 = 5/18$

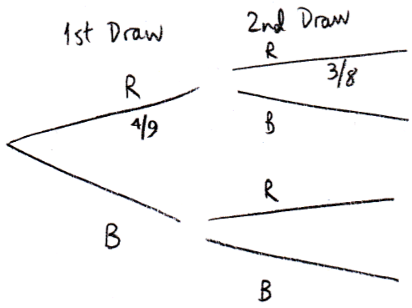
(i) Probability of drawing 2 reds $= \frac{1}{6}$

(ii) Probability of drawing at least 1 blue

$= P(R,B) + P(B,R) + P(B,B) = \frac{5}{18} + \frac{5}{18} + \frac{5}{18} = \frac{5}{6}$

The pupil's working was correct, but rather a lot of time was wasted in calculating the probabilities for part ii).

It was only necessary to realise that "at least one blue card" is the same as "*not* two red cards" and subtract the result from i).



$R,R = \frac{\cancel{4}^{1}}{\cancel{9}_{3}} \times \frac{\cancel{3}^{1}}{\cancel{8}_{2}} = \frac{1}{6}$

(i) Probability of drawing 2 reds $= \frac{1}{6}$

(ii) Probability of drawing at least 1 blue

= "Probability of anything <u>except</u> 2 reds"

$1 - P(2\text{ reds}) = 1 - \frac{1}{6} = \frac{5}{6}$

Example (67): Harry has been investigating two coins for bias by tossing them together and noting the combinations. The following results occurred after 100 trials ;

Two heads: 26   Two tails: 27   A head and a tail: 47

Harry reckoned that the coins must be biased, as there were three combinations to test and that the results should have been closer to 33 for each combination.

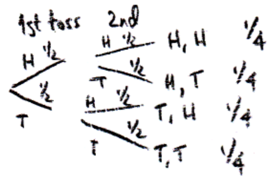
i) Is Harry's reasoning correct here ? Explain.
ii) From Harry's results, can the coins be classed as fair ? Explain.

(i) Harry's reasoning is incorrect, as 100 coin tosses is too small a number. He should extend the experiment to 200 tosses.
(ii) There are three possible outcomes, i.e. 'two heads', 'two tails', 'a head and a tail.' Coins are therefore unlikely to be fair.

The pupil is wrong on both counts here, but a tree diagram would have helped in answering part ii).

The correct answer with full explanation is shown below :

(i)



Harry's reasoning is incorrect
There are two ways of making 'a head and a tail' but only one way of making 'two heads' and only one way of making two tails.

∴ P(H,H) = ¼ ; P(H,T) + P(T,H) = ½ ; P(T,T) = ¼

(ii) Expected results from (i) : Two heads: ¼ of 100 = 25
Two tails: ¼ of 100 = 25
Head & Tail ½ of 100 = 50

Results are close enough to theoretical to pass coins as fair.