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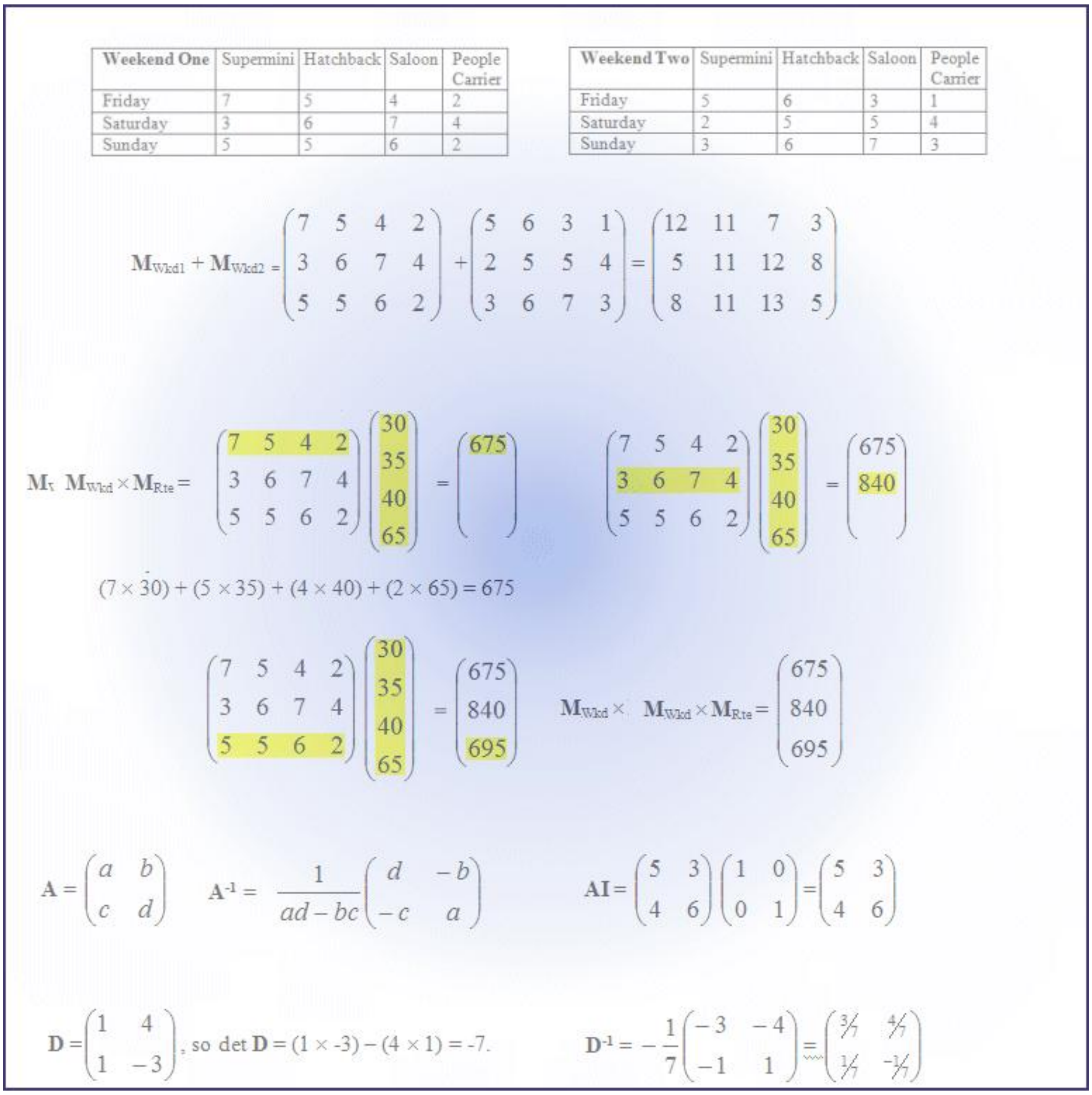
M.K. HOME TUITION

Mathematics Revision Guides
Level: International GCSE

MATRICES



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MATRICES

Practical Introduction.

A car-hire firm records its fleet usage for one weekend as follows, by day and type:

Weekend One	Supermini	Hatchback	Saloon	People Carrier
Friday	7	5	4	2
Saturday	3	6	7	4
Sunday	5	5	6	2

The values can be expressed in the form of a **matrix**, plural **matrices**. In mathematics, it is a specific term signifying a collection of numbers in an array, and not just a vogue word for a table of items.

You have probably come across matrices as column vectors in vector algebra, and in transformations.

Thus, the hirings for each day can be expressed as matrices as follows:

Friday: $\mathbf{M}_{\text{Fri}} = \begin{pmatrix} 7 & 5 & 4 & 2 \end{pmatrix}$

Saturday: $\mathbf{M}_{\text{Sat}} = \begin{pmatrix} 3 & 6 & 7 & 4 \end{pmatrix}$

Sunday: $\mathbf{M}_{\text{Sun}} = \begin{pmatrix} 5 & 5 & 6 & 2 \end{pmatrix}$.

Alternatively, all three days' hirings can be combined in a single matrix;

$$\mathbf{M}_{\text{Wkd}} =. \begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix}$$

Order of a matrix.

The order of a matrix is the number of rows and columns that it contains.
Matrices $\mathbf{M}_{\text{Fri}}$ to $\mathbf{M}_{\text{Sun}}$ each have one row and four columns, whilst matrix $\mathbf{M}_{\text{Wkd}}$ has three rows and four columns.

The order, $m \times n$, of a matrix is always quoted with the number of rows, m, first; thus the order of $\mathbf{M}_{\text{Fri}}$ is 1×4 and that of $\mathbf{M}_{\text{Wkd}}$ is 3×4.

A matrix with one row is a row matrix; a matrix with one column is a column matrix; a matrix with equal numbers of columns and rows is a square matrix.

Examples (1): Find the orders of the following matrices: i) $\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 \end{pmatrix}$; ii) $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}$;

iii) $\mathbf{C} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$; iv) $\mathbf{D} = \begin{pmatrix} 3 & 1 \\ -2 & 0 \\ 0 & 4 \\ 5 & 0 \end{pmatrix}$

i) Matrix **A** has one row and three columns – it is a 1×3 row matrix.
ii) Matrix **B** has two rows and two columns, so it is a 2×2 square matrix.
iii) Matrix **C** is a 2×1 column matrix.
iv) Matrix **D** is a 4×2 matrix.

Addition of matrices.

We can add together $\mathbf{M}_{Sat}$ and $\mathbf{M}_{Sun}$ to give the combined Saturday and Sunday car hirings.
To add two (or more) matrices, we add the corresponding elements.

$\mathbf{M}_{Sat} + \mathbf{M}_{Sun} = \begin{pmatrix} 3 & 6 & 7 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 5 & 6 & 2 \end{pmatrix}$.

$= \begin{pmatrix} 3+5 & 6+5 & 7+6 & 4+2 \end{pmatrix} = \begin{pmatrix} 8 & 11 & 13 & 6 \end{pmatrix}$.

$\mathbf{M}_{Sun} + \mathbf{M}_{Sat} = \begin{pmatrix} 5 & 5 & 6 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 7 & 4 \end{pmatrix}$.

$= \begin{pmatrix} 5+3 & 5+6 & 6+7 & 2+4 \end{pmatrix} = \begin{pmatrix} 8 & 11 & 13 & 6 \end{pmatrix}$.

Addition of matrices is commutative, (and also associative), exactly as addition of numbers.

Matrices can also be subtracted, thus

$\mathbf{M}_{Sat} - \mathbf{M}_{Sun} = \begin{pmatrix} 3 & 6 & 7 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 5 & 6 & 2 \end{pmatrix}$.

$= \begin{pmatrix} 3-5 & 6-5 & 7-6 & 4-2 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & 2 \end{pmatrix}$.

The result is not very meaningful here, except maybe to compare Saturday and Sunday hirings.

Supposing the car-hire firm decided to extend its range to include stretch limos the week after the earlier example results were published:

On the first weekend of the new schedule taking place, the takings were as follows:

Weekend Two	Supermini	Hatchback	Saloon	People Carrier	Stretch Limo
Friday	5	6	3	1	2
Saturday	2	5	5	4	1
Sunday	3	6	7	3	1

If we enter the results in a matrix, we obtain $\mathbf{M}_{Wkd2} = \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix}$.

The car-hire firm wants to combine the hirings for that particular weekend with those from the previous one, $\mathbf{M}_{Wkd}$. What goes wrong here, and how can it be corrected ?

If we were to find the sum of the two matrices, $\mathbf{M}_{Wkd} + \mathbf{M}_{Wkd2}$

$$\begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix}$$

we would run into trouble since there is nothing in the left-hand matrix to correspond with the fifth column in the right-hand one. In other words, the orders of the two matrices are different; $\mathbf{M}_{Wkd}$ is of order 3×4 but $\mathbf{M}_{Wkd2}$ is of order 3×5.

Since there were no stretch limos available in the previous week, we can modify $\mathbf{M}_{Wkd}$ by adding an extra column of zeros and making it into a 3 × 5 matrix $\mathbf{M}_{Wkd1}$.

We can now find the matrix sum:

$$\mathbf{M}_{Wkd1} + \mathbf{M}_{Wkd2} = \begin{pmatrix} 7 & 5 & 4 & 2 & 0 \\ 3 & 6 & 7 & 4 & 0 \\ 5 & 5 & 6 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 11 & 7 & 3 & 2 \\ 5 & 11 & 12 & 8 & 1 \\ 8 & 11 & 13 & 5 & 1 \end{pmatrix}$$

If two matrices have the same order, they are **conformable** for addition.

Example (2):.

If matrix **A** $\begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 1 & -4 \end{pmatrix}$ and matrix **B** = $\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -5 & -1 \end{pmatrix}$, find **A + B, B** + **A, A** – **B** and **B – A**.

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2-1 & 0+3 \\ 1+2 & (-2)+1 \\ 1-5 & (-4)-1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & -1 \\ -4 & -5 \end{pmatrix}$$

$$\mathbf{B} + \mathbf{A} = \begin{pmatrix} (-1)+2 & 3+0 \\ 2+1 & 1-2 \\ -5+1 & (-1)-4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & -1 \\ -4 & -5 \end{pmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2-(-1) & 0-3 \\ 1-2 & (-2)-1 \\ 1-(-5) & (-4)-(-1) \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -1 & -3 \\ 6 & -3 \end{pmatrix}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} (-1)-2 & 3-0 \\ 2-1 & 1-(-2) \\ -5-1 & (-1)-(-4) \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 1 & 3 \\ -6 & 3 \end{pmatrix}$$

Just as with numbers, addition of matrices is commutative, but subtraction isn't !
This is expected, since we are adding or subtracting corresponding elements from each matrix.

Example (3):.

ii) Given matrices **A** = $\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$, **B** = $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ and **C** = $\begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix}$,

find (**A** + **B**) + **C** , **A** + (**B** + **C**), (**A** - **B**) - **C** and **A** - (**B** - **C**).

(**A** + **B**) + **C** = $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 5 & -3 \end{pmatrix}$.

A + (**B** + **C**) = $\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 5 & -3 \end{pmatrix}$.

(**A** - **B**) – **C** = $\begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} 4 & -3 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 9 \end{pmatrix}$.

A - (**B** - **C**) = $\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} -5 & 5 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ 3 & -1 \end{pmatrix}$

Again, as with numbers, addition of matrices is associative, but subtraction isn't !

Supposing that matrix **A** = $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ and matrix **Z** = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the sum **A** + **Z** = $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, which is the same as the original matrix **A**.

Matrix **Z** is one example of a zero (or null) matrix. When added to another conformable matrix **M**, the original matrix **M** is left unchanged.

Zero matrices can be of any order, e.g. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$.

Multiplication of Matrices.

This requires a little more practice.

We will return to the car-hire example for Weekend One:

Weekend One	Supermini	Hatchback	Saloon	People Carrier
Friday	7	5	4	2
Saturday	3	6	7	4
Sunday	5	5	6	2

The matrices for each day's hirings are:

$\mathbf{M}_{\text{Fri}} = \begin{pmatrix} 7 & 5 & 4 & 2 \end{pmatrix}$; $\mathbf{M}_{\text{Sat}} = \begin{pmatrix} 3 & 6 & 7 & 4 \end{pmatrix}$; $\mathbf{M}_{\text{Sun}} = \begin{pmatrix} 5 & 5 & 6 & 2 \end{pmatrix}$.

The matrix for the combined hirings is ;

$$\mathbf{M}_{\text{Wkd}} =. \begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix}$$

The company charges a daily rate of £30 for superminis, £35 for hatchbacks, £40 for saloons and £65 for people-carriers.

The rates can also be expressed as a matrix, but this time it will be a column matrix;

$\mathbf{M}_{\text{Rte}} = \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix}$. Its order is 4×1.

Given the data above, we can use matrix multiplication to work out the total value of all combined hirings for Friday, by multiplying matrices $\mathbf{M}_1$ and $\mathbf{M}_5$.

$$\mathbf{M}_{\text{Fri}} \times \mathbf{M}_{\text{Rte}} = \begin{pmatrix} 7 & 5 & 4 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix}.$$

To multiply the matrices, we must visualise $\mathbf{M}_{\text{Rte}}$ turned anticlockwise through 90°, and corresponding terms multiplied and finally added to give a single element.

$$\mathbf{M}_{\text{Fri}} \times \mathbf{M}_{\text{Rte}} = \left((7 \times 30) + (5 \times 35) + (4 \times 40) + (2 \times 65)\right) = \left(675\right).$$

Note how $\mathbf{M}_{\text{Fri}}$ is of order 1×4 and $\mathbf{M}_{\text{Rte}}$ is of order 4×1, whereas their product is of order 1×1.

The result is a rather trivial matrix with one row and one column, and its sole element represents the value of combined takings, in £, for all car hirings for Friday.

The first (and only) row of the left-hand matrix ($\mathbf{M}_1$) is combined with the first (and only) column of the right-hand matrix ($\mathbf{M}_5$).

To multiply together larger matrices, we still follow the same idea;

$$\mathbf{M}_{\text{Wkd}} \times \mathbf{M}_{\text{Rte}} = \begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix}.$$

Firstly we combine **row 1** of the left-hand matrix with **column 1** of the right-hand matrix:
(7 × 30) + (5 × 35) + (4 × 40) + (2 × 65) = 675.

$$\begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix} = \begin{pmatrix} 675 \\ \\ \end{pmatrix}$$

∴ The element in row 1, column 1 of the resulting matrix is 675 (the Friday takings in £).

Next we combine **row 2** of the left-hand matrix with **column 1** of the right-hand matrix:
(3 × 30) + (6 × 35) + (7 × 40) + (4 × 65) = 840.

$$\begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix} = \begin{pmatrix} 675 \\ 840 \\ \end{pmatrix}$$

∴ The element in row 2, column 2 of the resulting matrix is 840 (the Saturday takings in £).

Finally we combine **row 3** of the left-hand matrix with the **column 1** of the right-hand matrix:
(5 × 30) + (5 × 35) + (6 × 40) + (2 × 65) = 695.

$$\begin{pmatrix} 7 & 5 & 4 & 2 \\ 3 & 6 & 7 & 4 \\ 5 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix} = \begin{pmatrix} 675 \\ 840 \\ 695 \end{pmatrix}$$

∴ The element in row 3, column 1 of the resulting matrix is 695 (the Sunday takings in £).

$$\mathbf{M}_{\text{Wkd}} \times \mathbf{M}_{\text{Rte}} = \begin{pmatrix} 675 \\ 840 \\ 695 \end{pmatrix}.$$

Note how $\mathbf{M}_{\text{Wkd}}$ is of order 3 × 4 and $\mathbf{M}_{\text{Rte}}$ is of order 4 × 1, whereas their product is of order 3 × 1.
Also, note how $\mathbf{M}_{\text{Wkd}}$ has the same number of columns as $\mathbf{M}_{\text{Rte}}$ has rows.

- **The summed combination of row *m* of the left-hand matrix with column *n* of the right-hand matrix gives the element in row *m* and column *n* of the product.**

- **The product of two matrices will therefore have the same number of rows as the left-hand one and the same number of columns as the right-hand one. In other words the product of a matrix of order $a \times b$ with one of order $c \times d$ gives a result of order $a \times d$.**

Returning to the hirings for Weekend Two:

Weekend Two	Supermini	Hatchback	Saloon	People Carrier	Stretch Limo
Friday	5	6	3	1	2
Saturday	2	5	5	4	1
Sunday	3	6	7	3	1

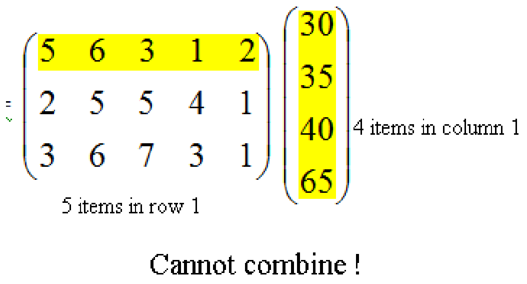
The matrix for the second weekend's details is $\mathbf{M}_{\text{Wkd2}} = \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix}$.

The car-hire firm also wants to work out the hire receipts for the second weekend, which includes the stretch limo hirings.

What goes wrong when trying to find $\mathbf{M}_{\text{Wkd2}} \times \mathbf{M}_{\text{Rte}}$, and how must we correct the problem , given that the hire charge for the stretch limo is £100 ?

$$\mathbf{M}_{\text{Wkd2}} \times \mathbf{M}_{\text{Rte}} = \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \end{pmatrix}.$$

We try to combine **row 1** of the left-hand matrix with **column 1** of the right-hand matrix:
$(5 \times 30) + (6 \times 35) + (3 \times 40) + (1 \times 65) + (2 \times \text{what?})$



The hirings matrix $\mathbf{M}_{\text{Wkd2}}$ has 5 columns but the rates matrix $\mathbf{M}_{\text{Rte}}$ has only 4 rows, leaving the entry for the stretch limo unmatched. The matrices cannot be multiplied in their present state.

The rates matrix is missing an entry for the hire charge of £100 for the stretch limo. It needs to be modified into the 5×1 matrix $\mathbf{M}_{\text{Rte2}} = \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \\ 100 \end{pmatrix}$.

Now we can find the matrix product: $\mathbf{M}_{\text{Wkd2}} \times \mathbf{M}_{\text{Rte2}} = \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \\ 100 \end{pmatrix}$

Combining **row 1** of the left-hand matrix with **column 1** of the right-hand matrix:
$(5 \times 30) + (6 \times 35) + (3 \times 40) + (1 \times 65) + (2 \times 100) = 745$.
$\therefore$ The element in row 1, column 1 of the resulting matrix is 745 (the Friday takings in £).

Combining **row 2** of the left-hand matrix with **column 1** of the right-hand matrix:
$(2 \times 30) + (5 \times 35) + (5 \times 40) + (4 \times 65) + (1 \times 100) = 795$.
$\therefore$ The element in row 2, column 2 of the resulting matrix is 795 (the Saturday takings in £).

Finally we combine **row 3** of the left-hand matrix with the **column 1** of the right-hand matrix:
$(3 \times 30) + (6 \times 35) + (7 \times 40) + (3 \times 65) + (1 \times 100) = 875$.
$\therefore$ The element in row 3, column 1 of the resulting matrix is 875 (the Sunday takings in £).

$$\mathbf{M}_{\text{Wkd2}} \times \mathbf{M}_{\text{Rte2}} = \begin{pmatrix} 5 & 6 & 3 & 1 & 2 \\ 2 & 5 & 5 & 4 & 1 \\ 3 & 6 & 7 & 3 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 35 \\ 40 \\ 65 \\ 100 \end{pmatrix} = \begin{pmatrix} 745 \\ 795 \\ 875 \end{pmatrix}.$$

- **Thus, for two matrices to be conformable for multiplication, the number of columns in the left-hand one must equal the number of rows in the right-hand one.**

Examples (4):

i) Find the matrix product **AB** where **A** = $\begin{pmatrix} 2 & 1 & 4 \end{pmatrix}$ and **B** = $\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$.

ii) Find the matrix product **AB** where **A** = $\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ and **B** = $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Does **BA** exist ?

i) Since **A** is a row matrix and **B** is a column matrix, their product **AB** has only one element.

Combining the only row of **A** with the only column of **B** gives
$(2 \times 3) + (1 \times 2) + (4 \times 7) = 36$.

Hence **AB** = $\begin{pmatrix} 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 36 \end{pmatrix}$.

ii) **A** = $\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ and **B** = $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Since **A** is a 2 × 2 matrix and **B** is a 2 × 1 matrix, their product **AB** will be a 2 × 1 matrix.

Row 1 of **A** and the only column of **B** combine to give $(1 \times 4) + (3 \times 1) = 7$.
∴ The element in row 1, column 1 of **AB** is 7.
Row 2 of **A** and the only column of **B** combine to give $(2 \times 4) + (3 \times 1) = 11$.
∴ The element in row 2, column 1 of **AB** is 11.

Row 1 of A / Col 1 of B

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ \ \end{pmatrix}$$

$(1 \times 4) + (3 \times 1) = 7$

Row 2 of A / Col 1 of B

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$$

$(2 \times 4) + (3 \times 1) = 11$

Hence **AB** = $\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$.

Because **B** has one column and **A** has two rows, the product **BA** = $\begin{pmatrix} 4 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ cannot exist.

(The column count in the left-hand matrix is unequal to the row count in the right-hand one.)

Example (5):

Let matrices **A** = $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ and **B** = $\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$. Find the products **AB** and **BA**. What do you notice ?

Since both **A** and **B** are 2 × 2 square matrices, their product **AB** is also of the same order.

AB = $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ 14 & 28 \end{pmatrix}$ (Working shown below).

Row 1 of A / Col 1 of B

$$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & \\ & \end{pmatrix}$$

$(3\times2) + (1\times3) = 9$

Row 1 of A / Col 2 of B

$$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ & \end{pmatrix}$$

$(3\times5) + (1\times4) = 19$

Row 2 of A / Col 1 of B

$$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ 14 & \end{pmatrix}$$

$(4\times2) + (2\times3) = 14$

Row 2 of A / Col 2 of B

$$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ 14 & 28 \end{pmatrix}$$

$(4\times5) + (2\times4) = 28$

Using the same method, the product **BA** = $\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 26 & 12 \\ 25 & 11 \end{pmatrix}$.

The last result shows that **matrix multiplication is not generally commutative**, although some exceptions occur**.**

Let **A** = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, **B** = $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

AB = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

BA = $\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$

The terms in each product are clearly different.

Example (6):

Let matrices **A** = $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$, **B** = $\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ and **C** = $\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$.

Find the matrix products (**AB**)**C** and **A**(**BC**). what do you notice ?

AB = $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 19 \\ 14 & 28 \end{pmatrix}$ (from Example (5))

BC = $\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 11 \\ 8 & 6 \end{pmatrix}$

(**AB**)**C =** $\begin{pmatrix} 9 & 19 \\ 14 & 28 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 17 & 39 \\ 28 & 56 \end{pmatrix}$

A(**BC**) = $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 11 \\ 8 & 6 \end{pmatrix} = \begin{pmatrix} 17 & 39 \\ 28 & 56 \end{pmatrix}$

Matrix multiplication is associative.

Proof : (we will show it here for 2 × 2 matrices)

Let **A =** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, **B** = $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and **C** = $\begin{pmatrix} i & j \\ k & l \end{pmatrix}$

AB = $\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ and hence **(AB)C =** $\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix}$

$$= \begin{pmatrix} (ae+bg)i+(af+bh)k & (ae+bg)j+(af+bh)l \\ (ce+dg)i+(cf+dh)k & (ce+dg)j+(cf+dh)l \end{pmatrix}$$

$$= \begin{pmatrix} aei+bgi+afk+bhk & aej+bgj+afl+bhl \\ cei+dgi+cfk+dhk & cej+dgj+cfl+dhl \end{pmatrix}$$

Similarly **BC** = $\begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix}$ and hence **A(BC)** = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix}$

$$= \begin{pmatrix} a(ei+fk)+b(gi+hk) & a(ej+fl)+b(gj+hl) \\ c(ei+fk)+d(gi+hk) & c(ej+fl)+d(gj+hl) \end{pmatrix}$$

$$= \begin{pmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{pmatrix}$$

Checking terms, we see that **(AB)C = A(BC)** term for term.

Example (7):

Let matrices **A** = $\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix}$ and **B** = $\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}$. Find **AB** and **BA.**

AB = $\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 12 & 15 \end{pmatrix}$.

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & \\ & \end{pmatrix}$$

Row 1 of A / Col 1 of B

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ & \end{pmatrix}$$

Row 1 of A / Col 2 of B

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 12 & \end{pmatrix}$$

Row 2 of A / Col 1 of B

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 12 & 15 \end{pmatrix}$$

Row 2 of A / Col 2 of B

BA = $\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & 15 & -9 \\ 16 & 0 & -8 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & & \\ & & \\ & & \end{pmatrix}$$

Row 1 of B / Col 1 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & \\ & & \\ & & \end{pmatrix}$$

Row 1 of B / Col 2 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ & & \\ & & \end{pmatrix}$$

Row 1 of B / Col 3 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & & \\ & & \end{pmatrix}$$

Row 2 of B / Col 1 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & 15 & \\ & & \end{pmatrix}$$

Row 2 of B / Col 2 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & 15 & -9 \\ & & \end{pmatrix}$$

Row 2 of B / Col 3 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & 15 & -9 \\ 16 & & \end{pmatrix}$$

Row 3 of B / Col 1 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & 15 & -9 \\ 16 & 0 & \end{pmatrix}$$

Row 3 of B / Col 2 of A

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & -2 \\ 0 & -5 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -10 & 4 \\ 0 & 15 & -9 \\ 16 & 0 & -8 \end{pmatrix}$$

Row 3 of B / Col 3 of A

This time, changing the order of the multiplication has also changed the orders of the resulting matrices: **AB** is a 2×2 square matrix ; **BA** is a 3×3 one.

Example (8):

Let matrices $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find **AI** and **IA.** What do you notice ?

$$\mathbf{AI} = \begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix} \text{ and } \mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 4 & 6 \end{pmatrix}.$$

In each case the resulting matrix is equal to **A.**

The Unit Matrix (Identity Matrix).

The matrix **I** in Example (8) is an example of an **identity matrix** or **unit matrix**.
Unit matrices must always be square, and they are characterised by containing 1's along the main diagonal, running from the top left (row 1, column 1) and zeros in all the other entries (except the trivial 1 × 1 matrix $\mathbf{I}_1 = (1)$.

Other unit matrices are $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

A conformable matrix **A** can either be pre-multiplied by the unit matrix to give **IA** or post-multiplied by it to give **AI**. Either way, the matrix **A** remains unchanged.

Finding the Inverse Matrix – determinants, singular matrices.

Two matrices are inverses of each other if their product is the unit matrix, and by definition they must both be square.

The inverse of a 2 × 2 square matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given as $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

The value $ad - bc$ is known as the **determinant** of the matrix **A**, or det **A.**

Also $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. (another special case of commutativity)

Proof : $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$

Multiplying by $\frac{1}{ad - bc}$ gives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Similarly $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & bd - bd \\ -ac + ac & -bc + ad \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$

Multiplying by $\frac{1}{ad - bc}$ gives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Examples (9): Find the determinants and inverses of the following matrices:

i) **A** = $\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$; ii) **B** = $\begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix}$; iii) **C** = $\begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}$; iv) **D** = $\begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}$; v) **E** = $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

Something goes wrong with one of them – which one ?

i) If **A** = $\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$, then det **A** = (5 × 5) – (6 × 4) = 1.

Hence **A**$^{-1}$ = $\frac{1}{1}\begin{pmatrix} 5 & -4 \\ -6 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -6 & 5 \end{pmatrix}$.

ii) **B** = $\begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix}$, so det **B** = (-2 × 3) – (-1 × 5) = -1.

Hence **B**$^{-1}$ = $-\frac{1}{1}\begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$.

iii) **C** = $\begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}$, so det **C** = (4 × 1) – (0 × -2) = 4.

Hence **C**$^{-1}$ = $\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix}$.

iv) **D** = $\begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}$, so det **D** = (1 × -3) – (4 × 1) = -7.

Hence **D**$^{-1}$ = $-\frac{1}{7}\begin{pmatrix} -3 & -4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/7 & 4/7 \\ 1/7 & -1/7 \end{pmatrix}$.

v) **E** = $\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$, so det **E** = (2 × 3) – (6 × 1) = 0.

Hence **E**$^{-1}$ = $\frac{1}{0}\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}$. We must stop here though, because we cannot divide by zero.

If a square matrix has a determinant of zero, then it cannot have an inverse, and is therefore known as **singular**.

Note the fractions in the inverses of matrices **C** and **D** in the last example.

Application of matrices to solving linear simultaneous equations.

Supposing we want to solve the simultaneous equations

$x + 4y = 2$
$x - 3y = -5$

We could solve them by using the elimination method:
$x + 4y = 2$ *A*
$x - 3y = -5$ *B*
$7y = 7$ *A-B*

This gives $y = 1$, and so the value could be substituted into either of the original equations. Substituting into equation *A* gives $x + 4 = 2$, therefore $x = -2$.

The solution to these equations is therefore $x = -2$, $y = 1$.

We could also take an alternative approach using matrices by rewriting the original equations as $\begin{pmatrix} 1 & 4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$. The left-hand matrix is the same as **D** in part iv) of Example (9), and its inverse $\mathbf{D}^{-1}$ was worked out as $\begin{pmatrix} 3/7 & 4/7 \\ 1/7 & -1/7 \end{pmatrix}$.

We could then pre-multiply both sides of the equation by the inverse matrix $\mathbf{D}^{-1}$:

$$\mathbf{D}^{-1}\mathbf{D}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{D}^{-1}\begin{pmatrix} 2 \\ -5 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{D}^{-1}\begin{pmatrix} 2 \\ -5 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/7 & 4/7 \\ 1/7 & -1/7 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

remembering that $\mathbf{D}^{-1}\mathbf{D} = \mathbf{D}\mathbf{D}^{-1} =$ the identity matrix **I.**

Hence $x = -2$, $y = -1$.

Example (10): Use the matrix method and the result from Example (9) (i) to solve the simultaneous equations $5x + 4y = 26$; $6x + 5y = 32$.

The matrix **A** in Example 4 i) is $\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$ and its inverse $\mathbf{A}^{-1} = \begin{pmatrix} 5 & -4 \\ -6 & 5 \end{pmatrix}$.
The simultaneous equations in matrix form are

$$\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 26 \\ 32 \end{pmatrix}, \text{ or } \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 26 \\ 32 \end{pmatrix}.$$

Pre-multiplying both sides by $\mathbf{A}^{-1}$ we have $\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 26 \\ 32 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} 26 \\ 32 \end{pmatrix}$

$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ The solution is $x = 2$, $y = 4$.

Powers of Matrices.

A matrix can be multiplied by itself, thus **AA** = $\mathbf{A}^2$, and we can continue with higher powers.

Example (11): Let matrix **P** = $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find $\mathbf{P}^2$ and $\mathbf{P}^3$.

$\mathbf{P}^2 = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$; $\mathbf{P}^3 = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 41 & 84 \\ 42 & 83 \end{pmatrix}$;

Example(12): Let matrix **Q =** $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$. Find $\mathbf{Q}^2$ and $\mathbf{Q}^3$.

$\mathbf{Q}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 8 & 9 \end{pmatrix}$; $\mathbf{Q}^3 = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 26 & 27 \end{pmatrix}$;

Q is a triangular matrix, in that the element(s) on one side of the leading diagonal are zeros, and all other elements non-zero.

Notice how the zero element remains equal to zero each time we multiply by **Q**. Also, the leading diagonal entries of $\mathbf{Q}^2$ and $\mathbf{Q}^3$ correspond to the square and the cube of 1 and 3 respectively.
The element in the lower left is not as easy to work out though.

Example(13): Let matrix **R =** $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. Find $\mathbf{R}^2$ and $\mathbf{R}^3$.

$\mathbf{R}^2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$; $\mathbf{R}^3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & 8 \end{pmatrix}$;

R is a diagonal matrix, in that all elements away from the leading diagonal are zeros.
This time, only the elements along the leading diagonal remain non-zero when we take the powers $\mathbf{Q}^2$ and $\mathbf{Q}^3$, and they correspond to the square and the cube of 3 and 2 respectively.

In general, for any diagonal matrix, **D** = $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\mathbf{D}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$.