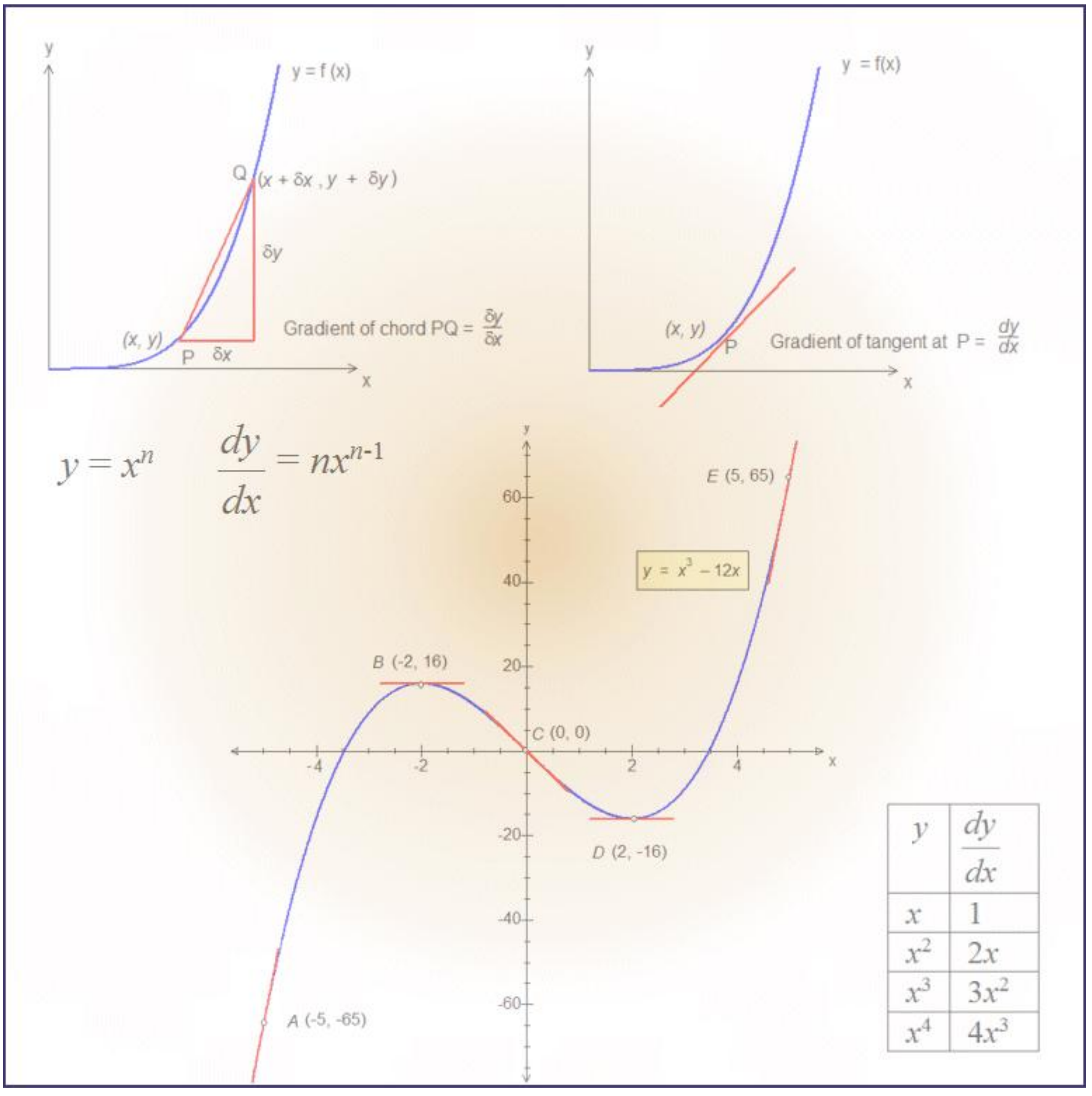
M.K. HOME TUITION

Mathematics Revision Guides
Level: IGCSE

DIFFERENTIATION



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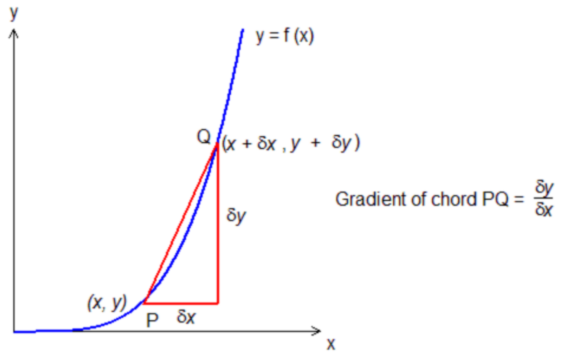
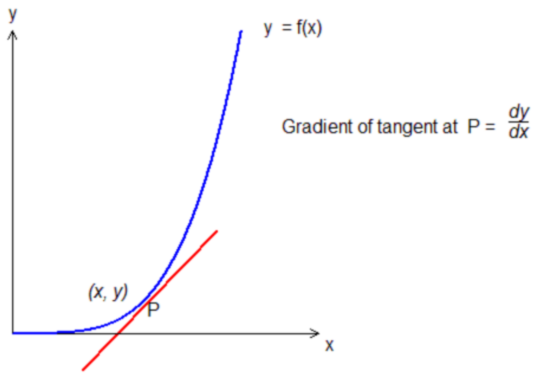
DIFFERENTIATION.

The gradient of a curve.

The idea of a gradient was brought about when studying linear functions. Now, linear functions have a constant gradient. The function $y = 2x - 5$, for example, has a gradient of 2 regardless of the value of x.

The gradient of a curve, by contrast, changes continuously along its length.





Take the chord PQ in the upper diagram. Its gradient is given by $\dfrac{\delta y}{\delta x}$ (delta y over delta x) where δx is a small change in x and δy is the corresponding change in y.

As the chord PQ becomes smaller and smaller, the ratio $\dfrac{\delta y}{\delta x}$ approximates to the gradient of the curve at P more and more closely, until the points P and Q coincide and the chord becomes a tangent at P.

As Q moves towards P, the value of $\delta x \to 0$ and $\dfrac{\delta y}{\delta x} \to \dfrac{dy}{dx}$.

$\dfrac{dy}{dx}$ is the **gradient function** and represents the **derivative** of y with respect to x.

Note that the terms δy , δx, dy and dx are not products – the symbols δ and d mean "difference in".

Note also that although $\dfrac{\delta y}{\delta x}$ is a **number**, $\dfrac{dy}{dx}$ is a **function.**

Some gradient functions can be worked out using the method on the previous page – known as **differentiation from first principles**.

Example (1): Differentiate $y = x^2$ from first principles.

Going back to the diagram on page 2, if we set $y = x^2$, then a small change in x (here δx) will cause a corresponding change in y, namely δy.

Since $y = x^2$, it follows that $y + \delta y = (x + \delta x)^2$.

$\therefore\ y + \delta y = x^2 + 2x(\delta x) + (\delta x)^2$.

Subtracting the original function gives $\delta y = 2x(\delta x) + (\delta x)^2$, and dividing throughout by δx, we have

$$\frac{\delta y}{\delta x} = 2x + \delta x .$$

As δx tends to zero, $\frac{\delta y}{\delta x} \to \frac{dy}{dx} = 2x$.

$\therefore$ $2x$ is the **derived function** of x^2.

The derived function of a polynomial.

The method of differentiation from first principles was just a demonstration – we have standard rules to work out gradient functions far more rapidly than that !

Any polynomial function $y = x^n$, where n is a constant, has a gradient function of

$$\frac{dy}{dx} = nx^{n-1}$$

In other words, you multiply by the power, and then reduce the power by 1.

Original function y	Derived function, $\frac{dy}{dx}$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Also, note the following:

The derivative of a constant function is zero.

If a function is multiplied by a constant, then its derivative is multiplied by the same constant.
For example, the derivative of x^2 is $2x$, so the derivative of $5x^2$ is $10x$.

If a function consists of separate terms added together, then the derivative of the sum is the sum of the derivatives of the separate terms.
For example, the derivative of $3x^3 - x^2$ is $9x^2 - 2x$.

Examples (2): Find the derived function of i) $2x$; ii) 8; iii) $4x^2$; iv) x^7; v) $2x^3 + 7x^2 - 5x + 4$.

i) The derivative of $2x$ is 2 (remember $x = x^1$).

ii) The derivative of 8 is 0 (8 is a constant – the derivative of any constant function is 0).

iii) The derivative of $4x^2$ is $8x$. (the result is 4 times $2x$, the derivative of x^2).

iv) The derivative of x^7 is $7x^6$ (multiply by the power, here 7, and reduce 7 by 1.)

v) The derivative of $2x^3 + 7x^2 - 5x + 4$ is $6x^2 + 14x - 5$ (each term's derivative summed together).

Example (3): Find the gradient of the curve $y = 4x^3$ at the point (2, 32).

Here, $\frac{dy}{dx} = 12x^2$, so the gradient at the point (2, 32) is 12×2^2 or 48.

Example (4): Find the coordinates of the point on the curve $y = 2x^2 - 3x - 7$ where the gradient is 5.

The gradient, $\frac{dy}{dx}$, is $4x - 3$, and so we solve $4x - 3 = 5$ to find the x-coordinate of the required point, namely 2. Substituting $x = 2$ into the original function gives the full coordinates of (2, -5).

Fractional and negative powers.

The rule of finding derivatives of polynomials can also be applied to fractional and negative powers.

Examples (5): Differentiate i) $\sqrt{x}$; ii) $\frac{1}{x}$; iii) $\frac{1}{x^2}$; iv) $\sqrt[3]{x}$.

i) Firstly, rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$, and then apply the rule of multiplying by the power and reducing the power by 1.

The derivative is thus $\frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$
(recall the laws of indices on fractional and negative powers).

ii) The expression can be rewritten as $y = x^{-1}$ and by applying the usual rule, the gradient function is $(-1)x^{-2}$ or $-\frac{1}{x^2}$.

iii) Since $\frac{1}{x^2} = x^{-2}$, differentiation gives $-2x^{-3} = -\frac{2}{x^3}$.

iv) Rewriting $\sqrt[3]{x}$ as $x^{\frac{1}{3}}$, the standard rule gives a derivative of $\frac{1}{3}x^{-\frac{2}{3}}$ or $\frac{1}{3x^{\frac{2}{3}}}$.

Example (6): Find the gradient of the curve $y = \sqrt{x}$ at the point (16, 4)

From Example (5) i) , $y = x^{\frac{1}{2}}$ has a derived function of $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$.

$\therefore$ At (16, 4), the gradient is therefore $\frac{1}{2\sqrt{16}}$ or $\frac{1}{8}$.

Example (7): Find the gradient of the curve $y = \frac{1}{x}$ at the point $(5, \frac{1}{5})$

From Example (5) ii) $y = \frac{1}{x}$ has a derivative of $\frac{dy}{dx} = (-1)x^{-2}$ or $\frac{-1}{x^2}$.

$\therefore$ At $(5, \frac{1}{5})$ the gradient is $-\frac{1}{25}$.

Derivatives in function notation.

If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ (f dash x)

If $y = kf(x)$ where k is a constant, then $\frac{dy}{dx} = kf'(x)$.

Another way of saying this is $\frac{d}{dx}\left(k\left(f(x)\right)\right) = k\frac{d}{dx}(f(x))$.

If $y = f(x) \pm g(x)$ where $f(x)$ and $g(x)$ are separate functions of x, then $\frac{dy}{dx} = f'(x) \pm g'(x)$

Example (8): If $f(x) = x^3 - 7x + 4$, find $f'(x)$ and $f'(2)$

$f'(x) = 3x^2 - 7$, and therefore $f'(2) = (3 \times 2^2) - 7 = 5$.
Sometimes a function needs to be expressed in the right form before it can be differentiated.

Example (9): Differentiate $f(x) = (2x - 7)(x + 4)$

A product cannot be differentiated term by term, and so it must first be expanded into a form that can.
$(2x - 7)(x + 4) = 2x^2 + x - 28$, which can be differentiated to give $f'(x) = 4x + 1$.

Example (10): Differentiate $f(x) = \frac{x^4 + 1}{x^2}$

A quotient cannot be differentiated term by term, so it must be rewritten as

$$\frac{x^4 + 1}{x^2} = x^2 + \frac{1}{x^2}$$

Both terms can now be differentiated to give $f'(x) = 2x - \frac{2}{x^3}$.

Note that $\frac{1}{x^2} = x^{-2}$ and differentiation gives $\frac{-2}{x^3} = -2x^{-3}$.

(There are rules for differentiating products and quotients, but they are outside the scope of IGCSE.)

The following example returns to the ideas behind differentiation from first principles.

Example(11): A curve passes through points *P* (2, 17) and *Q* (2.1, 18.23).

i) Find the gradient of the chord *PQ*.

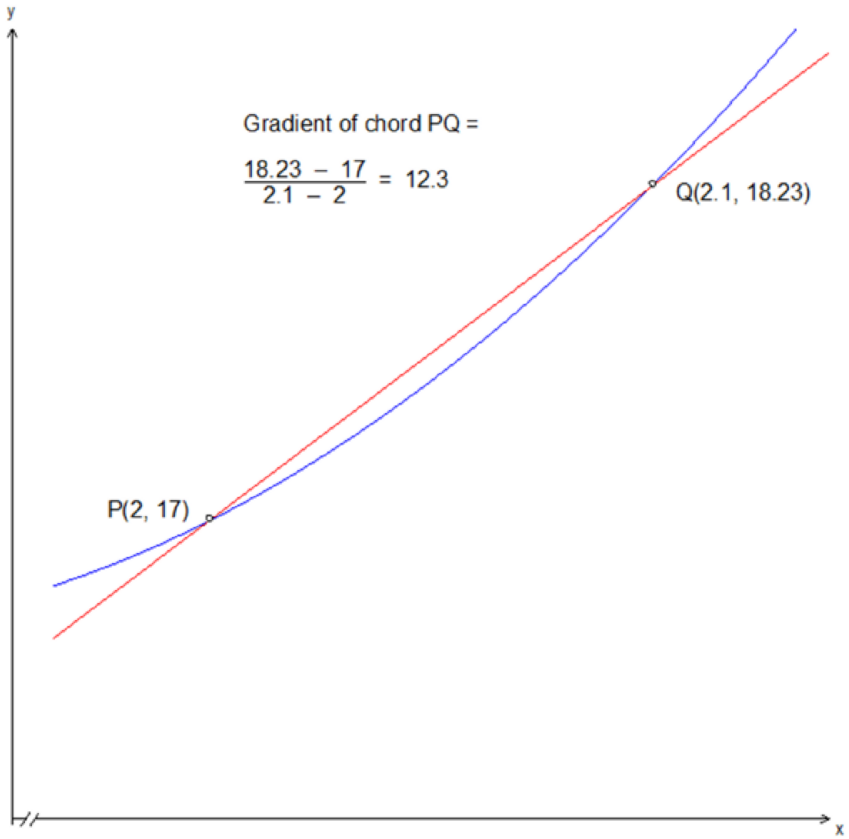
ii) Point *Q* is then moved ten times closer to *P*, to the point (2.01, 17.1203) . What is the gradient of the chord *PQ* now ?

iii) Suggest what happens to the chord and its gradient as *Q* is moved ever closer to *P*.

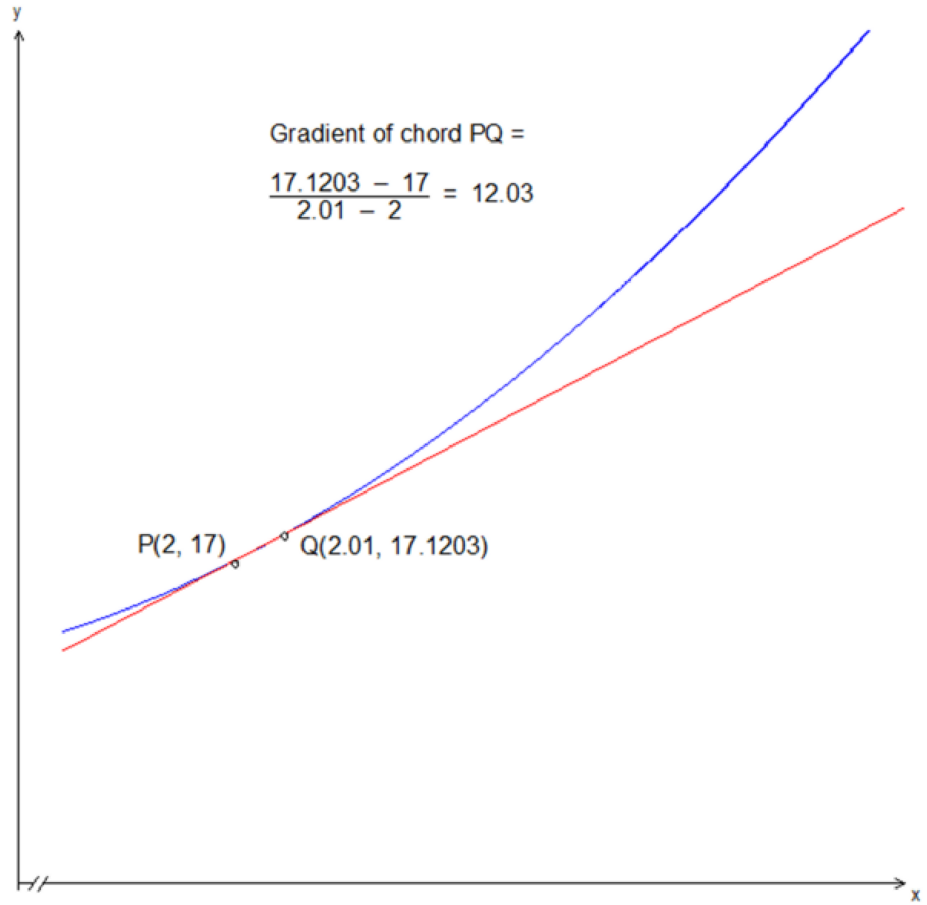
iv) The curve has an equation of $y = ax^2 + b$ where *a* and *b* are constants. Find the equation of the curve.

i) The chord *PQ* has a gradient of 12.3. (See diagram below – deliberately exaggerated).

Notice how the gradient of the chord at *P* is not particularly close to that of the tangent at that point.



ii) After point Q is moved to (2,01, 17.1203), the gradient of the chord PQ is 12.03.
Also, the chord is a much closer approximation to the tangent at point P now that PQ is ten times smaller.



iii) The gradient at P has changed from 12.3 to 12.03 as Q has become closer to P. Also, the chord approaches the tangent to P more closely the smaller the length of PQ.

This suggests that, when P and Q coincide, the chord PQ becomes a tangent at P and the gradient tends to a value of 12.

iv) The curve has an equation of $y = ax^2 + b$, and so its derivative $\frac{dy}{dx} = 2ax$.

From iii), $\frac{dy}{dx} = 12$ when $x = 2$, so $2a = 6$ and thus $a = 3$.

From this we can work out that the curve has an equation of $y = 3x^2 + b$.

To find b, substitute $x = 2$, $y = 17$ $\rightarrow$ $3x^2 + b = 17$ $\rightarrow$ $12 + b = 17$ $\rightarrow$ $b = 5$.

The equation of the curve is therefore $\mathbf{y = 3x^2 + 5}$.

Turning Points - Maxima and Minima.

A graph of a function of x is said to have a turning point (or points) if its derivative can take a value of zero for some value(s) of x. Turning points are also known as stationary points.

Turning points of quadratic graphs.

All quadratic graphs have one turning point, i.e. one occurrence of a zero derivative.

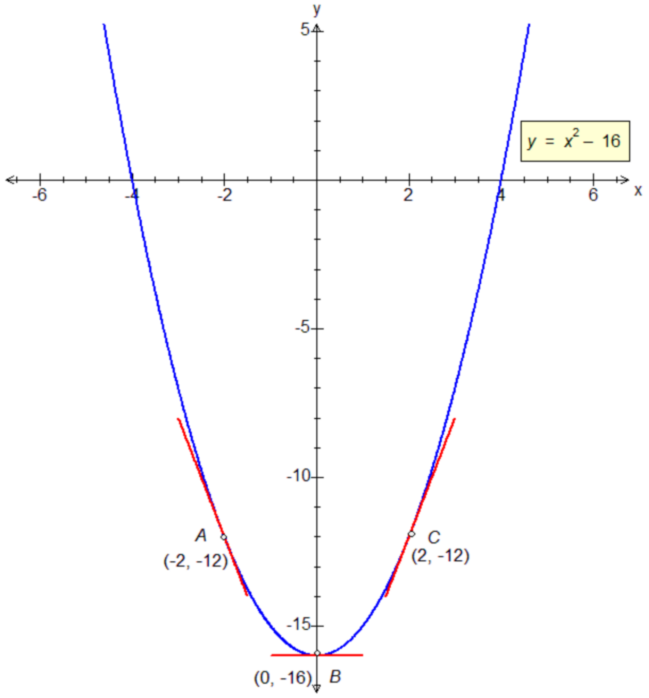
The graph of $y = x^2 - 16$ is shown on the right, and points A (-2, -12), B (0, -16) and C (2, -12) marked for reference.

At point A, the value of y is -12 and continues to decrease until it reaches a stationary minimum of -16 at point B. Moving from B to C and beyond, the value of y then begins to increase.

Between A and B, the gradient is negative but decreases to zero by the time we reach B. Thereafter, the gradient becomes increasingly positive.

Note the following:
The gradient at B is zero, so point B is a turning point – in fact it is a minimum point as y cannot be any lower.

The gradient of the curve goes from negative through zero to positive as x increases.



The following graph is that of $y = 4 - x^2$, and this time we are interested in the points A(-2, 0), B (0, 4) and C (2,0)

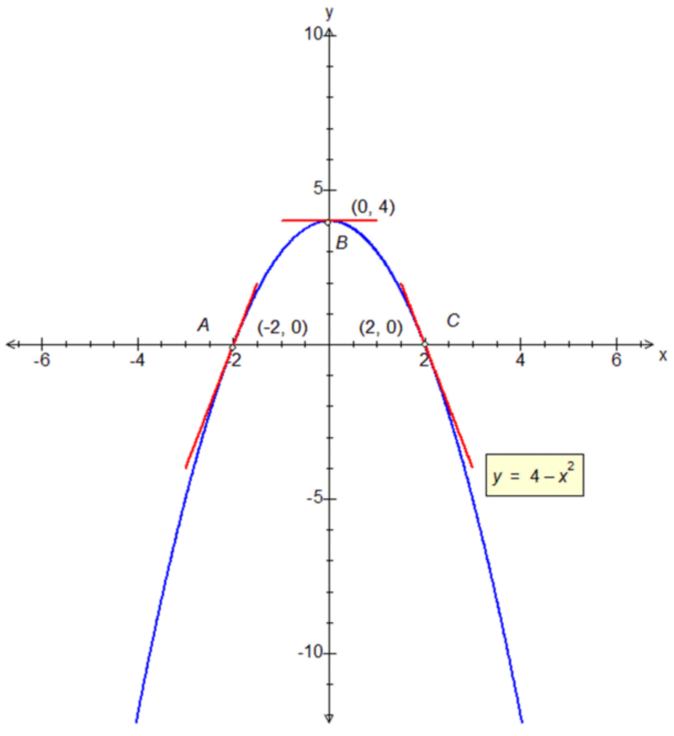
At point A, the value of y is 0 and continues to increase until it reaches a stationary maximum of 4 at point B.

Moving from B to C and beyond, the value of y then begins to decrease.

Between A and B, the gradient is positive but decreases to zero by the time we reach B. Thereafter, the gradient becomes increasingly negative.

The gradient at B is zero, so point B is a turning point – in fact it is a maximum point as y cannot be any higher.

The gradient of the curve goes from positive through zero to negative as x increases.



The last two examples illustrated the behaviour of a curve near a turning point:

- In the neighbourhood of a minimum point, the gradient changes from negative through zero to positive as x increases.

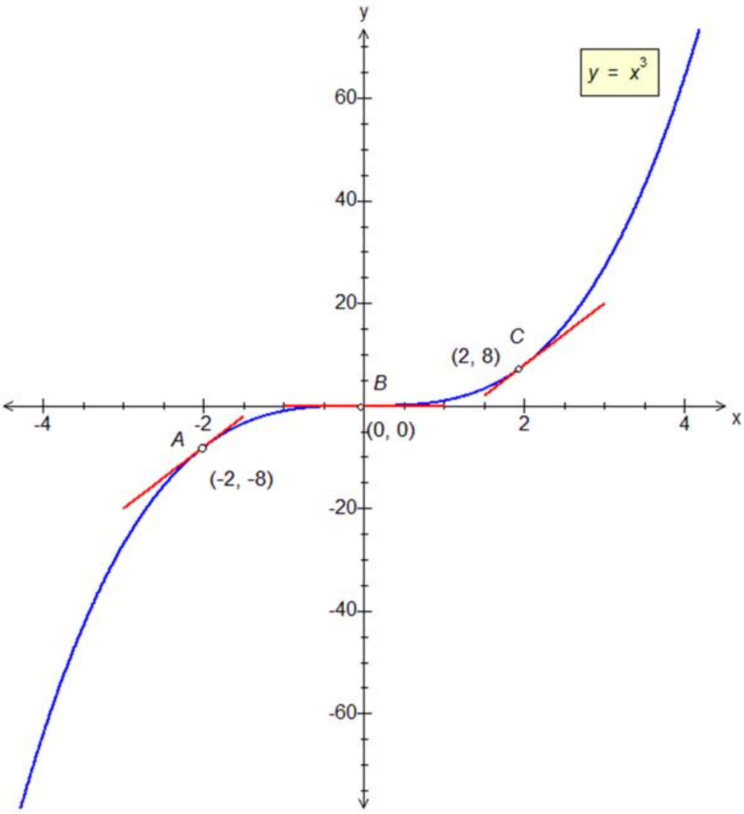
- In the neighbourhood of a maximum point, the gradient changes from positive through zero to negative as x increases.

The minimum and maximum points in the last example were also **global** since they were the only turning points.

Turning points of cubic graphs.

The derived function can give useful information about the behaviour of a curve. Two cubics are shown here.

Graph of $y = x^3$.



Points *A* (-2, -8), *B* (0, 0) and *C* (2, 8) are shown here.

For this graph, the value of y is never decreasing, changing from -8 at *A*, through zero at *B* and to 8 at *C*.

Between *A* and *B*, the gradient is positive but decreases to zero by the time we reach *B*. Thereafter, the gradient becomes positive again.

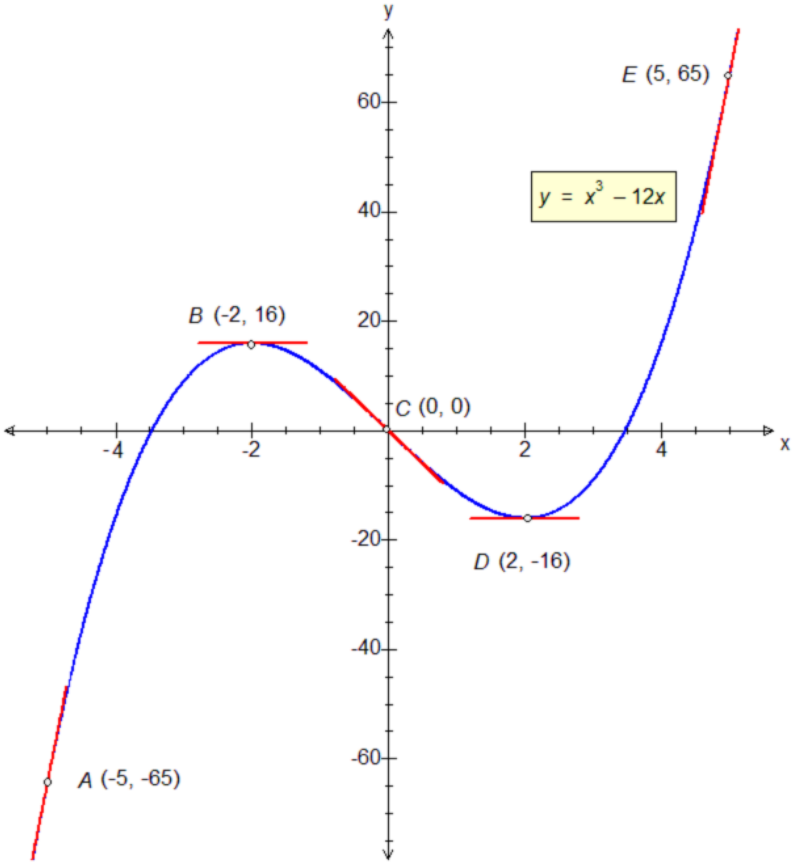
The gradient at *B* is zero, so point *B* is a turning point, but it is neither a minimum nor a maximum.

In fact it is a **stationary point of inflection.**

The sign of the gradient does not change on either side of the point of inflection.

(Points of inflection are not studied at IGCSE).

Graph of $y = x^3 - 12x$.



Something more interesting happens here !

The value of *y* increases from *A* to *B*, with a positive but decreasing gradient, until at *B* the gradient is zero and we have a stationary point – a **local maximum** at that point.

The value of *y* then decreases from *B* through *C*, with a negative gradient, until at *D* the gradient is zero again and we have another stationary point – this time a **local minimum**.

From *D* to *E*, the function is increasing again, with an increasing positive gradient.

Notice how the gradient behaves on either side of the local maximum at point *B*. It changes from positive through zero at *B* and then becomes negative.

The pattern is reversed in the case of the local minimum point *D*. There the gradient changes from negative through zero at *D* and then becomes positive.

Hence, as *x* increases:

In the neighbourhood of a local **minimum**, the gradient changes sign **from negative to positive**.
In the neighbourhood of a local **maximum**, the gradient changes sign **from positive to negative**.
In the neighbourhood of a point of **inflection**, the gradient **does not change sign**.

The maximum point at *B* is referred to as a **local** maximum because the value of *y*, namely 16, at that point is not an absolute maximum. At point *E*, for instance, for instance, $y = 65$.

For the same reason, the minimum at *D* is local only; there $y = -16$, but at point *A*, $y = -65$. *x* increases and the graph of the function goes beyond point *D*, the value of the function increases indefinitely beyond its value at *B*.

The Second Derivative.

Recalling the function notation for derivatives, we have

If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ (f dash x)

Differentiating $y = f(x)$ a second time gives the second derivative, written as

$\frac{d^2y}{dx^2} = f''(x)$ (f double dash x)

The second derivative can give information about the nature of any stationary points.
At a stationary point:

If $f''(x) > 0$, then the point is a local minimum.
If $f''(x) < 0$, then the point is a local maximum.

What if $f''(x) = 0$?

If $f''(x) = 0$ the result is inconclusive. It could mean a point of inflection (not covered in more detail), but could still be a maximum or a minimum. In that case, we must check the behaviour of the gradient in the neighbourhood of the stationary point, by choosing suitable values of x on each side of it.

If $f'(x)$ changes sign from -ve through 0 to +ve in the neighbourhood of the stationary point as x increases, then the point is a local minimum.

If $f'(x)$ changes sign from +ve through 0 to -ve in the neighbourhood of the stationary point as x increases, then the point is a local maximum.

Example (12) illustrates the method.

Example (12):

i) Show algebraically that the curve $f(x) = x^3 - 9x^2 + 15x – 5$ has a turning point at (5, -30) and another at (1,2).

ii) Determine which of those points is a maximum, and which is a minimum.

i) Firstly, we differentiate : $f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5)$.

Substituting $x = 5$ into the **equation of the curve**, we have $f(5) = 125 – 225 + 135 - 5 = -30$.
Likewise, substituting $x = 5$ into the **equation of the derivative**, we have $f'(5) = 75 – 90 + 15 = 0$.

$\therefore$ $f(x) = x^3 - 9x^2 + 15x – 5$ has a stationary point at (5, -30).

Next, substituting $x = 1$ into the original equation, $f(1) = 1 – 9 + 15 - 5 = 2$.
Finally, substituting $x = 5$ into the derivative gives $f'(1) = 3 – 18 + 15 = 0$.

$\therefore$ The curve also has a stationary point at (1,2).

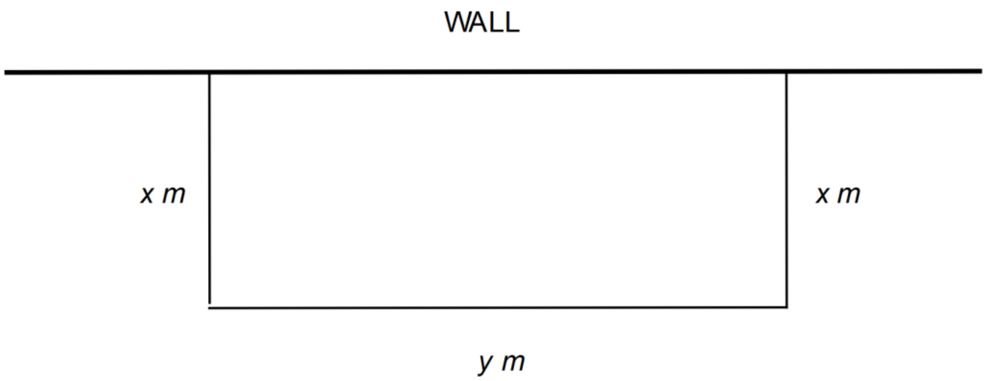
ii) We take the second derivative to distinguish between the two stationary points in i).

We differentiate $f'(x) = 3x^2 - 18x + 15$ to obtain $f''(x) = 6x – 18$.

When $x = 5$, $6x – 18 = 12$. The second derivative is positive here, and so (5, -30) is a local **minimum**.
When $x = 1$, $6x – 18 = -12$. The second derivative is negative here, and so (1, 2) is a local **maximum**.

Example (13):

A farmer wants to make a rectangular enclosure y metres long and x metres deep. There is a fixed solid wall on one side, and he has 32 metres of fencing available for the other three sides.
He wants to make the enclosed area as large as possible.



i) Show that the enclosed area, A, of the pen can be expressed as $A = x(32 - 2x)$ m^2.

ii) For $A = x(32 - 2x)$, find $\frac{dA}{dx}$, and hence find the value of x for which A takes a maximum value.

iii) Calculate this maximum value of A. Explain algebraically why it is a maximum.

i) The total length of the fencing is 32 m, so the length of the fence, $y = 32 - 2x$ m, and therefore the area $A = xy = x(32 - 2x)$ m^2.

ii) $A = x(32 - 2x) \rightarrow A = 32x - 2x^2$, hence $\frac{dA}{dx} = 32 - 4x$.

The maximum value of A occurs when this derivative is zero : $32 - 4x = 0$, i.e. when $x = 8$.

iii) When $x = 8$, $A = 256 - 128 = 128$.

$\therefore$ The maximum area enclosed by the fencing is 128 square metres, and that occurs when the depth of the fencing is 8 metres and the length 16 metres.

This area is indeed a maximum because the second derivative, $\frac{d^2A}{dx^2} = -4$ (i.e. < 0).

Example (14): Find the stationary points, if any, of the curve $f(x) = 1 - x^4$. Which, if any, is a maximum or a minimum ?

Differentiating, we have $f'(x) = -4x^3$.
Differentiating again, we have $f''(x) = -12x^2$.

At a stationary point, $f'(x) = 0$, so we solve $-4x^3 = 0$. The only solution is $x = 0$, so the only stationary point of the curve $f(x) = 1 - x^4$ is (0, 1).

To determine if (0, 1) is a maximum or a minimum, we take the second derivative, $f''(x) = -12x^2$. Unfortunately, this second derivative is also zero, so the test does not help.

We therefore need to find the gradient of the curve at two other points, one on each side of the stationary point. Two suitable values are at $x = -1$ and $x = 1$, i.e. at points (-1,0) and (1,0).

The gradient at (-1,0) is $f'(-1) = 4$; the corresponding gradient at (1, 0) is $f'(1) = -4$.

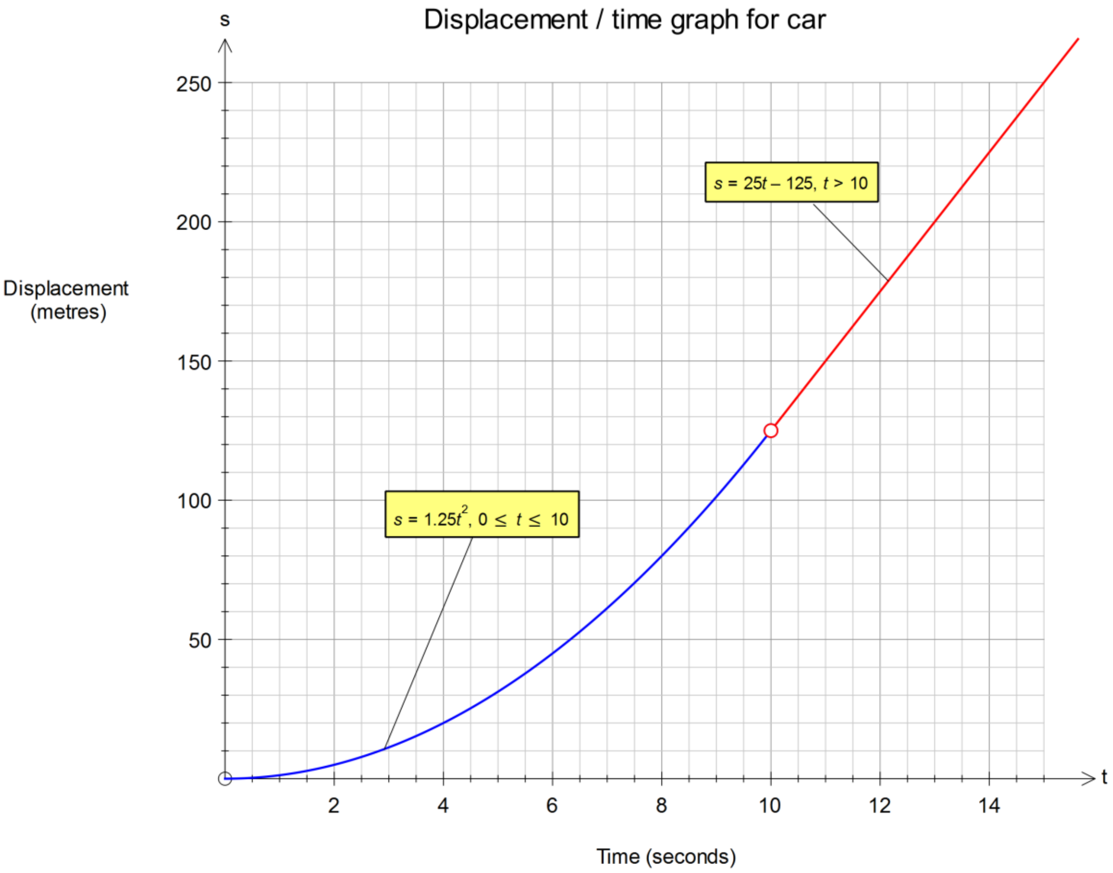
Therefore, as *x increases* from -1 through 0 to 1, the gradient changes sign from positive (4), through 0 to negative (-4) on either side of the stationary point at (0, 1).

Therefore the point (0, 1) is a local maximum.

Applications in Mechanics.

In the sections on real-life graphs, and especially travel graphs, we have seen the relationship between quantities such as displacement, velocity and acceleration.

A car is being driven along a track, and the graph below illustrates its displacement (distance from the starting point) in metres for the first 15 seconds.



For the first ten seconds, the graph's equation is a quadratic, $s = 1.25t^2$, but after that, it becomes linear, with the equation $s = 25t - 125$.

We can use derivative notation to illustrate the relationships between displacement, s, and velocity, v.

Because velocity is the rate of change of displacement with respect to time, t, we say $v = \frac{ds}{dt}$.

We can therefore find expressions for the velocity of the car by differentiation.

When $s = 1.25t^2$, $v = \frac{ds}{dt} = 2.5t$ (for the first 10 seconds).

When $s = 25t - 125$, $v = \frac{ds}{dt} = 25$ (thereafter) .

For the first ten seconds, the velocity (in m/s) varies depending on the time; after that, it is constant.
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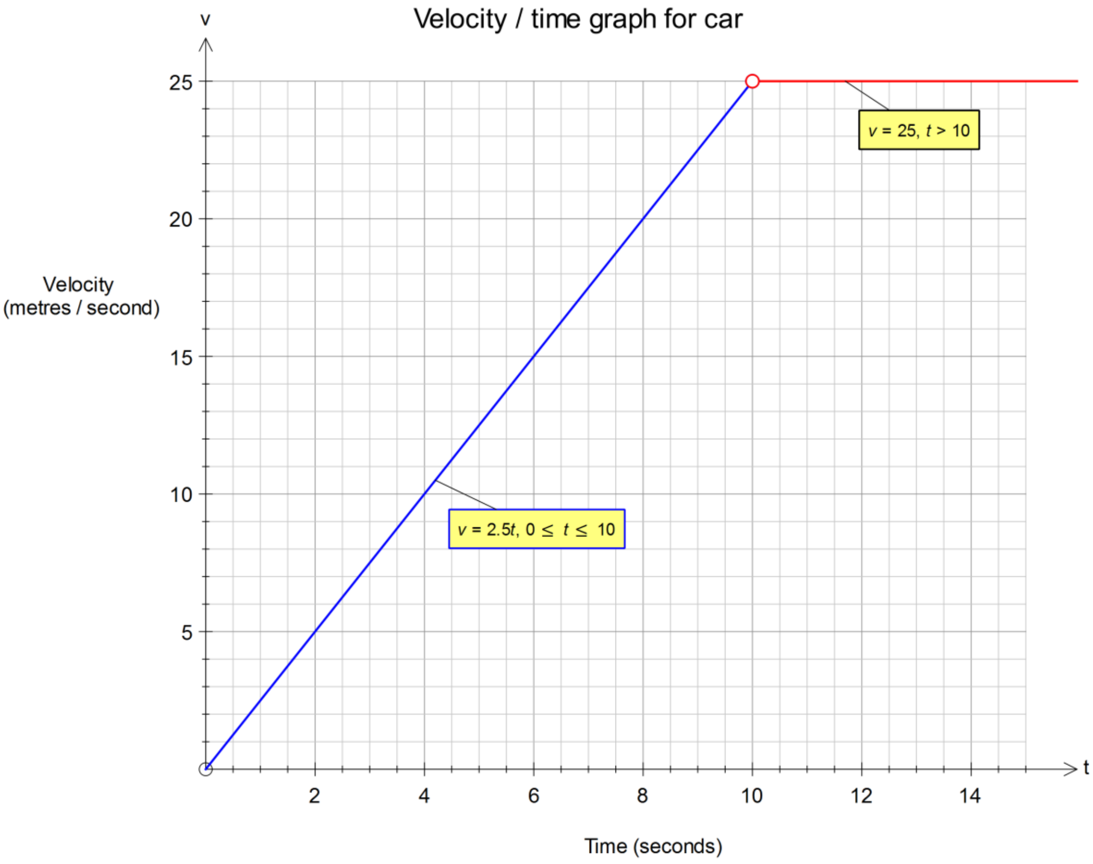
The corresponding velocity-time graph is shown below.



It can be seen that the velocity increases at a linear rate of 2.5 m/s per second for the first ten seconds, and then remains at 25 m/s afterwards.

In other words, the car is accelerating at a rate of 2.5 m/s per second for the first ten seconds, after which the acceleration falls to zero.

The rate of change of velocity (with respect to time) is also the rate of acceleration, so we also say $a = \frac{dv}{dt}$.

Differentiating the velocity expressions we have:

When $v = 2.5t$, $a = \frac{dv}{dt} = 2.5$ (for the first 10 seconds).

When $v = 25$, $a = \frac{dv}{dt} = 0$ (thereafter) .

Since $v = \frac{ds}{dt}$, we can also say $a = \frac{d^2s}{dt^2}$, i.e. we differentiate the displacement function once to find the velocity, and twice to find the acceleration.

Example (15): A particle moves in a straight line which passes through the fixed point *O.*

The particle's displacement, *s*, from *O* is given by $s = 12t^2 - 2t^3$
where *t* is the time in seconds and $0 \le t \le 6$.

i) Find an expression for the velocity of the particle in metres per second at time *t* seconds.

ii) Find the particle's displacement when $t = 4$, and show that this value is a maximum.

iii) At what time does the particle have zero acceleration ?

i) We differentiate *s* to find the velocity ; $v = \frac{ds}{dt} = 24t - 6t^2$.

ii) When $t = 4$, $s = (12 \times 16) - (2 \times 64) = 64$, i.e. the particle is 64 metres from *O*.

Also, $v = (24 \times 4) - (6 \times 16) = 96 - 96 = 0$, so the particle's velocity is zero at $t = 4$.
Differentiating again, $a = \frac{d^2s}{dt^2} = 24 - 12t$, and when $t = 4$, $a = -24$.

This second derivative is negative, so the particle's displacement takes a maximum value at 4 seconds.

iii) Since $a = 24 - 12t$ from ii), we solve $24 - 12t = 0$, giving $t = 2$.
$\therefore$ The particle has zero acceleration after 2 seconds.

Example (16): A ball is released into the air at a velocity of 25 m/s, from an initial height of 2 m.

Its height is given by the formula $h = 2 + 25t - 5t^2$, where t is the time elapsed in seconds.

(Ignore the actual dimensions of the ball, i.e. treat it as a particle.)

i) Find expressions for the velocity and acceleration of the ball.

ii) Find the height and velocity of the ball after 4 seconds. Explain the latter result.

iii) Find the maximum height attained by the ball, to the nearest metre.

iv) Use the quadratic formula to show that the ball falls back to the ground after just over 5 seconds .

i) The velocity of the ball is $v = \frac{dh}{dt} = 25 - 10t$ m/s.

The acceleration is $a = \frac{d^2h}{dt^2} = \frac{dv}{dt}$ -10 m/s^2.

ii) When $t = 4$, $h = 2+100-80 = 22$, so the ball is 22m above ground level after 4 seconds.

The velocity, v , = 25 - 40 or -15 m/s. The context of the question makes it clear that the *upwards* direction is *positive* , therefore the negative velocity signifies a *downwards* direction.

iii) The ball reaches its maximum height when $v = \frac{dh}{dt} = 0$, i.e. when $25 - 10t = 0$.

This is when $t = 2.5$ seconds. Substituting $t = 2.5$ into the height formula gives $h = 2 + 62.5 - 31.25$, or 33. The maximum height reached by the ball is thus 33 metres.

iv) We need to solve $h = 0$, namely $2 + 25t - 5t^2 = 0$. We use the general quadratic formula

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, or $t = \frac{-25 \pm \sqrt{625 + 40}}{-10}$. The solutions are -0.39 and 5.08.

Only the positive result is applicable here, so the ball falls back to the ground after 5.1 seconds.